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THEORY OF FLUORESCENT IRRADIANCE FIELDS IN LAKES AND SEAS

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PREFACE

This technical memorandum preserves an essentially complete manuscript which was found among the papers of Dr. Rudolph Preisendorfer after his untimely death. No editing of the manuscript has been done, other than a careful proofreading by Dr. Curtis Mobley.

ERRATA

Page	Line	Error
36	20	equation (8.3a) should be numbered (8.2a)
53	14	"$\rho(y,\pm\|j,j)$" should read "$\rho(y,\mp\|j,j)$"
60	18	"radiance" should read "irradiance"
61	10,14,15,16	"$R(y,-,i,j)$" should read "$R(y,-\|i,j)$"
81	12	"$\underline{H}(x,\pm,i)$" should read "$H(x,\pm,i)$"
136	1	"$\underline{H}_\eta$" should read "$\underline{\mathit{H}}_\eta$"
163	10	"$\underline{\mathit{R}}(x,z)$" should read "$R(x,z)$"
217	13,15	"$a(y,j)$" should read "$\bar{a}(y,j)$"
242	1	"$a_j(y,\lambda)$" should read "$a_j(\lambda)$"

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PART IV. INVERSE SOLUTION OF THE IRRADIANCE MODEL

THEORY OF FLUORESCENT IRRADIANCE FIELDS IN LAKES AND SEAS

Rudolph W. Preisendorfer

ABSTRACT. It is shown how to determine the irradiance field in lakes and seas that have fluorescing stratified layers of chlorophyll and other organic material. This is the direct solution of the irradiance field which starts from the depth-distribution of optical properties, in particular the spectral absorption and scattering functions of the material. Conversely, it is shown how to determine these optical properties, from irradiance probe measurements *in situ*, by inverting the direct solutions for the irradiance field. The present work forms the basis for applications of radiative transfer theory to remote sensing of seas and lakes and specifically for optically-based chlorophyll assays within such media. In particular, it is shown how to determine the intrinsic (or specific) optical properties of a natural hydrosol from irradiance measurements in the hydrosol.

1. INTRODUCTION

The theory of monochromatic radiative transfer in natural hydrosols is here extended to the heterochromatic case, with particular applications in mind for the optical assays of chlorophyll and related organic substances in lakes and seas. The present heterochromatic theory is designed to describe the spectral behavior of fluorescent processes in near-surface water layers containing chlorophyll-bearing plankton. These plankton stand at the base of the food chain in natural hydrosols. It will be shown how, knowing the spectral signatures of chlorophyll fluorescence and various other hydrosol ingredients, as viewed from within these layers, one can infer from optical measurements the concentrations of these materials present in the hydrosols. By determining the spectral reflectance of the medium as seen from the atmosphere, some applications to remotely sensed natural media are also possible. In general, with the present theory, which helps sort out the appearance of the spectral signatures as seen through intervening layers of water and the water surface itself, one can apply the principles of hydrologic optics to the problem of determining certain key physical and biological constituents of natural waters.

A. *Background*

Two recent works, namely Gordon (1979) and Spitzer and Wernand (1983), form the immediate point of departure for the present study. In Gordon (1979) it was shown that it is possible to explain the observed enhancement of the reflectance of the sea in the wavelength region around 685 nm by the presence of fluorescing chlorophyll-a in the near surface layers. This was established by setting up the equation of transfer for transpectral (inelastic) scatter, solving it in the first-order scattering approximation, and using the solution to compute the irradiance reflectance just beneath the surface. In this study we shall present exact numerical procedures to determine the irradiance reflectance just below and just above the surface. The work of Spitzer and Wernand applies the exact two-flow *monochromatic* irradiance model (as defined in H.O., Vol. V, sec. 8.4),* with a source term, to describe the fluorescence contribution to upward and downward irradiances at each wavelength for each depth in the water column. They developed a remote sensing procedure for rhodamine B concentrations that occur on the order of 1 $mg \cdot m^{-3}$ in near-surface layers. In the present study, the two-flow monochromatic irradiance model is generalized to the heterochromatic case. As a result the source term in the monochromatic case disappears, and the resulting equations are homogeneous, i.e., source free. This permits a simpler formalism for the solutions of the direct and inverse problems.

* 'H.O.' is short for the hydrologic optics reference, Preisendorfer, 1976b.

B. *Overview*

In the present work, we begin with the derivation of the equation of transfer for unpolarized radiance in which the volume scattering function accounts for scattering of photons with change of wavelength. The derivation is patterned after the procedure in Preisendorfer (1965, p. 63-66) using the volume transpectral scattering function (loc. cit., p. 59). In the derivation, care is taken to identify the various losses and gains of radiant energy due to absorption and scattering processes (cf. loc. cit. p. 60). As a result we obtain a general form of the equation of radiance transfer (cf. (8.1), below) with a new form of divergence relation for irradiance (cf. (9.5)) that includes the effects of transpectral scattering, true emission, and true absorption.

There are some other advances in the theory of the light field in lakes and seas in the present work that may be noted here. First, the theory of directly observable light fields (cf. e.g. Preisendorfer, 1961; or H.O., vol. V, sec. 9.2) is extended to the heterochromatic case (sec. 12, below). Moreover, virtually all of the material from sec. 15 onward is a new treatment of radiative transfer problems in natural hydrosols beyond that given in H.O., the main new features being (a) the elevation of all two-flow model equations in H.O. from the scalar (monochromatic) level to the vector (heterochromatic) level and (b) the inclusion of source terms in all the statements of the global interaction principles and their logical descendants (union rules, imbed rules and Riccatian quartets). These innovations, incidentally, are drawn directly from the applications of invariant imbedding procedures to water wave transport theory, developed by the author during his stay at the Hawaii Institute of Geophysics in the period from 1971 to 1976. For a list of these works, see the bibliography in Preisendorfer, 1977b, the last of the series of 14 water wave transport studies. The Hawaiian work most directly relevant to the present one is Preisendorfer, 1976a.

Finally, the inverse solution procedure for the present heterochromatic theory of the two-flow irradiance model is given in Part IV. This generalizes the monochromatic inverse theory presented in Preisendorfer and Mobley (1984).

C. *Level of the Theory*

Radiative transfer theory is a predominantly phenomenological theory of the interactions of photons with electrons located on or near the surfaces of complex atoms and molecules making up the pure water, the solutes, and the suspensoids comprising natural waters. In particular we can set up a theory of fluorescence consistent with experimental optical procedures without having to know the detailed quantum electrodynamic (QED) processes of photon-electron interactions. (For a useful perspective of photon interactions with matter, see Feynman, 1986). Rather, by choice, the attenuation and scattering functions in the equation of transfer (8.1) are consistently determinable by optical procedures conducted *in situ*. Indeed, Part IV below is designed to show how all the optical properties needed in the heterochromatic two-flow model (11.19) can be found from *in situ* irradiance measurements alone. The basic ideas for this self-consistent two-way model were recently illustrated in the monochromatic case in Preisendorfer and Mobley (1984). However, for knowledgeable application of the concepts of the present model, it still is advisable to become familiar with at least the rudiments of classical molecular theory of fluorescent photosynthetic processes. For this one may consult, e.g., Pringsheim (1963), and the books by Clayton (1965, 1970, 1971).

D. *Acknowledgments*

Ryan Whitney assisted with the word processor; Gini Curl drew the figures.

PART I. RADIANCE MODEL

2. RADIOMETRIC PRELIMINARIES

We collect here the four main radiometric concepts needed in the present study. These are the radiance function N and its three *irradiance* relatives, namely the plane irradiance H, the scalar irradiance h, and the vector irradiance $\underline{H}$.

The operational definition of the (unpolarized) radiance $N(y,\underline{\xi},\lambda)$ at point y in the direction (unit vector) $\underline{\xi}$ of wavelength λ (in nanometer $\equiv$ nm) is shown in Fig. 1. A narrow tube of small solid angle opening $\Delta\Omega$ (steradian $\equiv$ sr) allows radiant flux (Watt $\equiv$ W $\equiv$ Joule second^{-1} $\equiv$ $J \cdot s^{-1}$) to be collected over a plane diffusing surface D of small area ΔA (in m^2) normal to $\underline{\xi}$. The flux passes through a wavelength filter F that transmits a residual amount of radiant flux of a specified wavelength λ and small bandwidth $\Delta\lambda$ (nm). The transmitted flux activates an electric signal in a photocell P. Let $\Delta P(y,\underline{\xi},\lambda)$ (in $W \cdot nm^{-1}$) be the amount of the recorded *spectral radiant flux* (i.e., radiant power). Then the empirical (spectral) *radiance* at y along unit vector $\underline{\xi}$ of wavelength λ is defined by writing

$$'N(y,\underline{\xi},\lambda)' \quad \text{for} \quad \Delta P(y,\underline{\xi},\lambda)/\Delta A \Delta\Omega \ . \tag{2.1}$$

The ideal radiance is obtained from this in the limit as ΔA, $\Delta\Omega$, and $\Delta\lambda$ approach zero, at y, along $\underline{\xi}$, and at λ, respectively. The mks units of N are $W \cdot m^{-2} \cdot sr^{-1} \cdot nm^{-1}$.

From the preceding definitions, we see that $N(y,\underline{\xi},\lambda)$ may be envisioned as the radiant flux carried by photons of wavelength λ (or frequency $\nu = c/\lambda$) streaming at the speed of light c across ΔA through a solid angle $\Delta\Omega$ about the direction $\underline{\xi}$ normal to ΔA. Each photon contains a small, definite amount

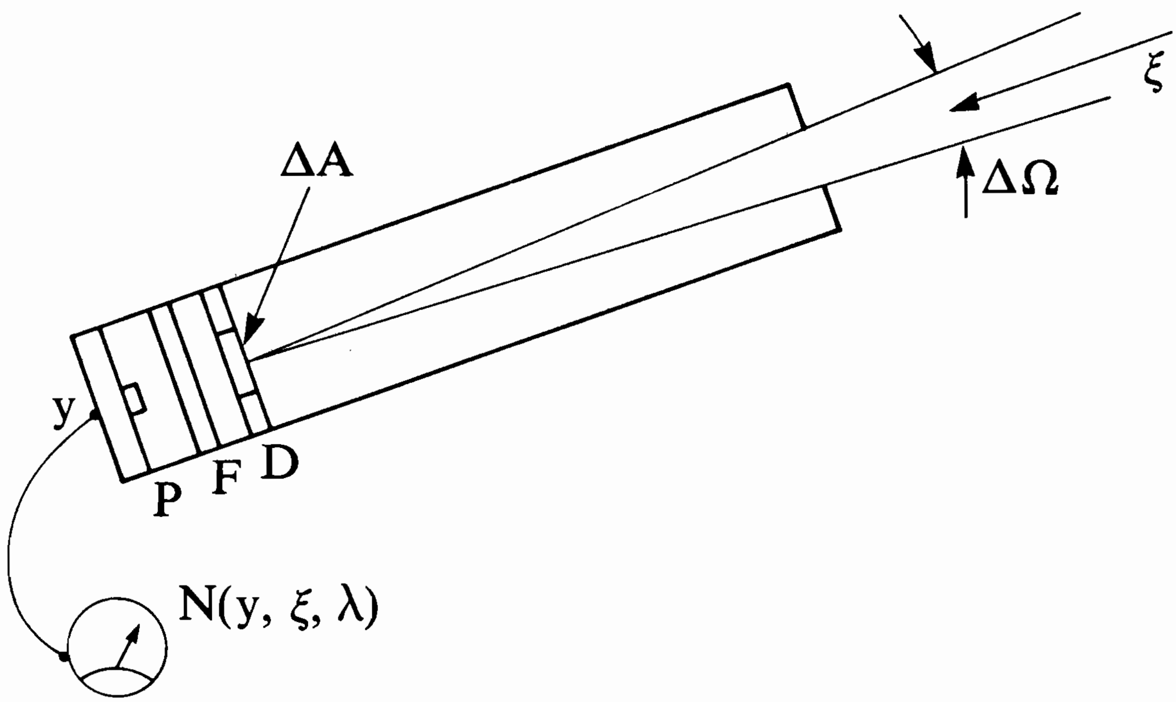

Figure 1--A meter used in the operational definition of the radiance $N(y,\underline{\xi},\lambda)$.

(i.e., a *quantum*) of energy of magnitude $h_o\nu = h_oc/\lambda$ J·photon^{-1} (of wavelength λ), where $h_o = 6.626 \times 10^{-34}$ J·s·photon^{-1} is Planck's constant (action per photon) and $c = 3.000 \times 10^8$ m·s^{-1}, with λ in nm. Let us write

$$'n(y,\underline{\xi},\lambda)' \quad \text{for} \quad N(y,\underline{\xi},\lambda)p(\lambda) \tag{2.2}$$

where we write

$$'p(y)' \text{ for } \lambda h_o/c \tag{2.2a}$$

$n(y,\underline{\xi},\lambda)$ is the *photon radiance* at y, along $\underline{\xi}$, of wavelength λ. The units of $n(y,\underline{\xi},\lambda)$ are photon·s^{-1}·m^{-2}·sr^{-1}·nm^{-1} when λ is in nanometers, say. Then $p(\lambda) \equiv \lambda/h_oc = 5.03 \times 10^{15}\ \lambda$ photon·J^{-1}. Thus $n(y,\underline{\xi},\lambda)$ is the *number* of *photons* of wavelength λ streaming across the collector, per second per square meter (of collector surface) per steradian (of collector acceptance cone) per nanometer (of spectrum wavelength scale). Note that photons of higher frequency (shorter wavelength) carry more energy (i.e., their quantum of energy h_oc/λ is larger) than those photons with lower frequency (longer wavelength). For instance, to generate the same radiance N, all else the same, one must have twice as many 700 nm photons as 350 nm photons. For example, the number of photons of wavelength 555 nm needed to generate 1 Watt per square meter on normal incidence, is 2.76×10^{18}.

The radiant flux at a point y generally streams in along all possible directions $\underline{\xi}$. The totality of directions $\underline{\xi}$ is the unit sphere Ξ (cf. Fig. 2). It is useful to partition Ξ into an upward hemisphere Ξ_+ and a lower hemisphere Ξ_- consisting, respectively, of all $\underline{\xi}$ such that $\underline{\xi}\cdot\underline{i}_3 > 0$ and $\underline{\xi}\cdot\underline{i}_3 < 0$ where $\underline{i}_3$ is the unit upward normal in a natural hydrosol. Then the *upward* (+) and *downward* (-) (spectral) *plane irradiances* at y are given by

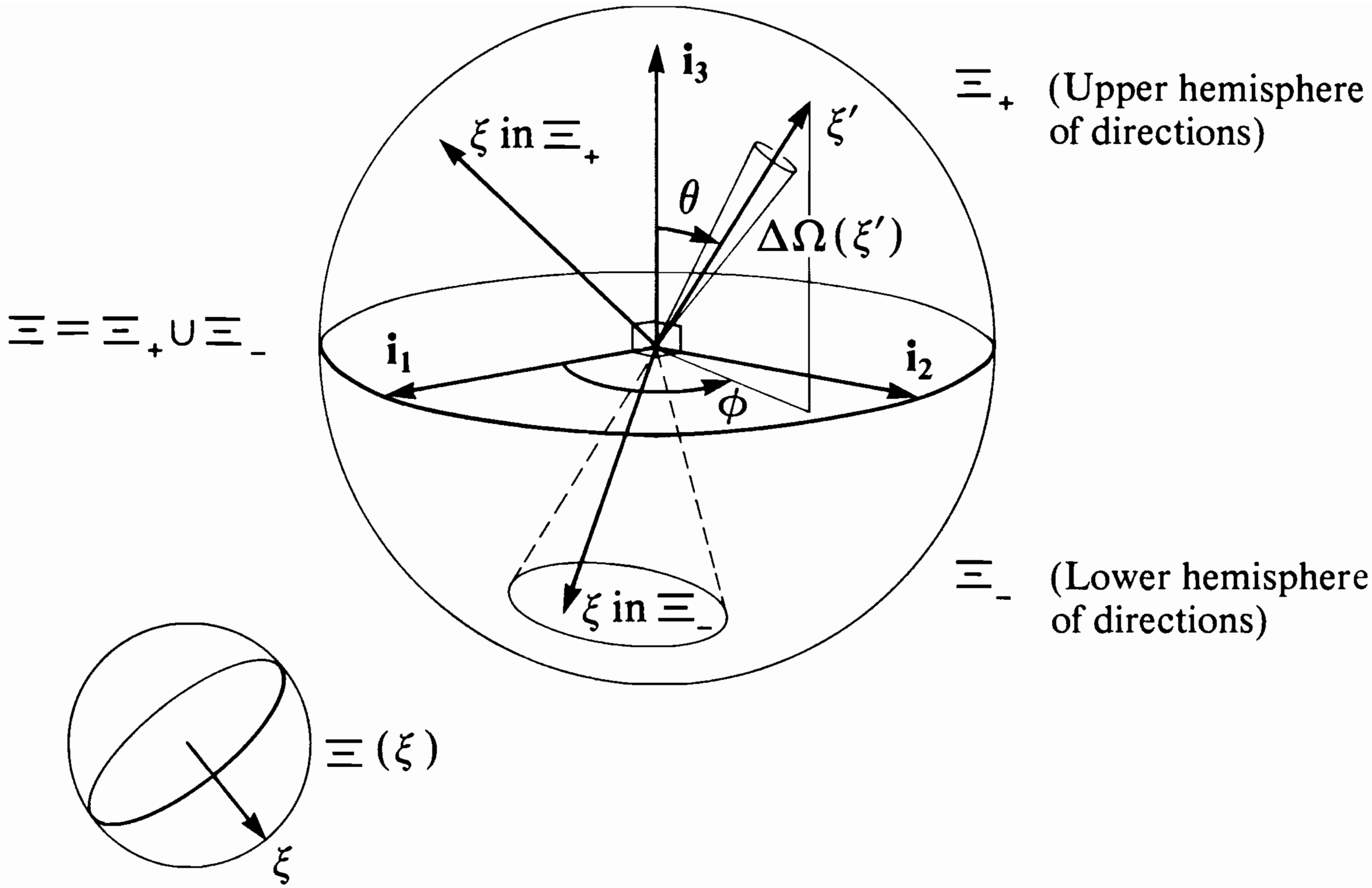

Figure 2--The unit sphere Ξ and associated quantities.

$$H(y,\pm,\lambda) \equiv \int_{\Xi_\pm} N(y,\underline{\xi}',\lambda)|\underline{\xi}'\cdot\underline{i}_3|\ d\Omega(\underline{\xi}') \qquad (W\cdot m^{-2}\cdot nm^{-1}) \qquad (2.3)$$

where $d\Omega(\xi')$ is an element of solid angle about direction $\underline{\xi}'$. In the spherical coordinate frame of Fig. 2 we have

$$d\Omega(\underline{\xi}) = \sin\theta\ d\theta\ d\phi \qquad (2.4)$$

and

$$\underline{\xi}\cdot\underline{i}_3 = \cos\theta\ (\equiv \mu) \qquad (2.5)$$

Hence $H(y,\pm,\lambda)$ in (2.3) may also be written

$$H(y,+,\lambda) = \int_0^{2\pi}\left[\int_0^{\frac{1}{2}\pi} N(y,\theta',\phi',\lambda)|\cos\theta'|\sin\theta' d\theta'\right]d\phi' \qquad (2.6)$$

$$= \int_0^{2\pi}\left[\int_0^{1} N(y,\mu',\phi',\lambda)|\mu'|d\mu'\right]d\phi' \qquad (2.7)$$

The expression for $H(y,-,\lambda)$ is similar (μ' now ranges from -1 to 0).

An operational definition of plane irradiance $H(y,\pm,\lambda)$ is given by means of the device shown in Fig. 3. A flat diffusing surface D of unit inward normal $\underline{n}$ and of small area ΔA at y sends radiant flux to a wavelength filter F which in turn transmits an amount $\Delta P(y,\underline{n},\lambda)$ of spectral radiant flux ($W\cdot nm^{-1}$) in the wavelength band $\Delta\lambda$ to a photocell P. Then we write

$$'H(y,\underline{n},\lambda)' \text{ for } \Delta P(y,\underline{n},\lambda)/\Delta A \qquad (W\cdot m^{-2}\cdot nm^{-1}) \qquad (2.8)$$

The collecting surface D in Fig. 3 may be arbitrarily oriented. When its inward normal $\underline{n}$ is $\underline{i}_3$, then we set $H(y,\underline{n},\lambda) = H(y,+,\lambda)$ and when $\underline{n}$ is $-\underline{i}_3$, then we set $H(y,-\underline{n},\lambda) = H(y,-,\lambda)$. The ideal surface in Fig. 3 collects amounts of

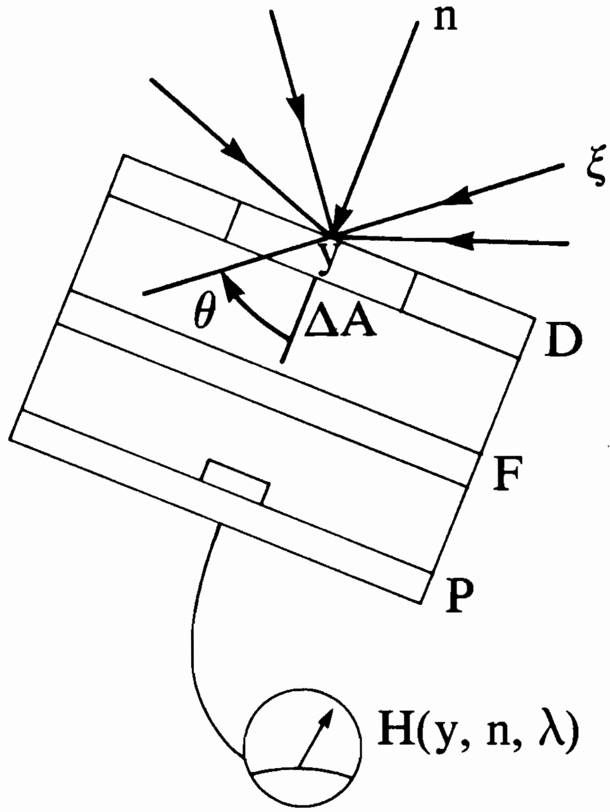

Figure 3--A meter used in the operational definition of the plane irradiance $H(y,\underline{n},\lambda)$.

radiant flux along $\underline{\xi}$ proportional to the amount of its projected area normal to $\underline{\xi}$. Such a surface is called a 'cosine collector', and when D is such a collector, then (2.8) is consistent with (2.3) and may be used to determine $H(y,\pm,\lambda)$ from an irradiance meter such as that indicated in Fig. 3.

Another type of irradiance that will be useful in the developments below is the *hemispherical (spectral) scalar irradiance* $h(y,\pm,\lambda)$ at y defined by

$$h(y,\pm,\lambda) \equiv \int_{\Xi_\pm} N(y,\underline{\xi}',\lambda)\, d\Omega(\underline{\xi}') \qquad (\mathrm{W\cdot m^{-2}\cdot nm^{-1}}) \qquad (2.9a)$$

and the sum of these, the *scalar irradiance*:

$$h(y,\lambda) \equiv \int_{\Xi} N(y,\underline{\xi}',\lambda)\, d\Omega(\underline{\xi}') \qquad (\mathrm{W\cdot m^{-2}\cdot nm^{-1}}) \qquad (2.9b)$$

$$= h(y,+,\lambda) + h(y,-,\lambda)$$

A device that in principle may be used to measure $h(y,-,\lambda)$ is schematically shown in Fig. 4 when $\underline{n} = -\underline{i}_3$. A spherical diffusing shell D collects radiant flux streaming in from all directions $\underline{\xi}'$ in Ξ_-. The environmental flux along directions in Ξ_+ to D is stopped by a shield SS which has a nonreflecting upper surface. The diffused transmitted flux within the cavity of D is funneled down to a filter F which passes only flux of wavelength λ in a band $\Delta\lambda$ on to a photocell P. If each small element of outer area of D is a cosine collector then the reading $h(y,-,\lambda)$ in Fig. 4 is, to within a fixed factor, proportional to that given in (2.9). In practice it is relatively difficult to accurately collect radiant flux in the way shown in Fig. 4. The better way is to measure $N(y,\underline{\xi}',\lambda)$ for a sufficient number of $\underline{\xi}'$ in Ξ_- and compute $h(y,\pm,\lambda)$ as shown in (2.9a). Fig. 4 merely serves to give an intuitive meaning to the integrations in (2.9a,b).

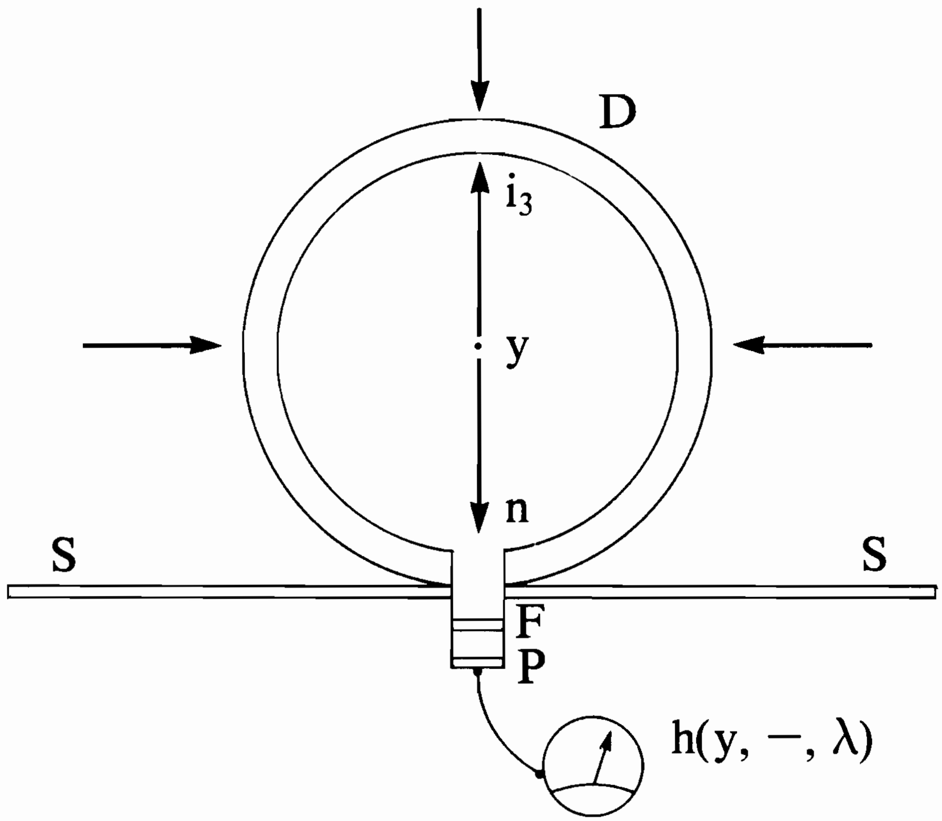

Figure 4--A hypothetical device used to measure the scalar irradiance $h(y,-,\lambda)$.

The final irradiance construct needed in the studies below is that of *vector irradiance* $\underline{H}(y,\lambda)$ defined by

$$\underline{H}(y,\lambda) \equiv \int_{\Xi} N(y,\underline{\xi}',\lambda)\ \underline{\xi}' d\Omega(\underline{\xi}') \qquad (\mathrm{W \cdot m^{-2} \cdot nm^{-1}}) \tag{2.10}$$

The intuitive meaning of the vector $\underline{H}(y,\lambda)$ may be discerned as follows. First we generalize (2.3) by writing

$$H(y,\underline{n},\lambda) \equiv \int_{\Xi_+(\underline{n})} N(y,\underline{\xi},\lambda)\ \underline{\xi}\cdot\underline{n}\ d\Omega(\underline{\xi}) \tag{2.11}$$

where

$$\Xi_+(\underline{n}) \equiv \{\underline{\xi}:\quad \underline{\xi}\cdot\underline{n} > 0\} \tag{2.12}$$

i.e., $\Xi_+(\underline{n})$ is the positive hemisphere of Ξ defined by the arbitrarily oriented unit inward normal $\underline{n}$ of an irradiance meter (Fig. 3) measuring $H(y,\underline{n},\lambda)$. Now, taking the inner product of $\underline{n}$ and the vector irradiance $\underline{H}(y,\lambda)$ in (2.10), we find by (2.11) that

$$\underline{n}\cdot\underline{H}(y,\lambda) = H(y,\underline{n},\lambda) - H(y,-\underline{n},\lambda) \tag{2.13}$$

$$\equiv \overline{H}(y,\underline{n},\lambda) \tag{2.14}$$

Hence knowing $\underline{H}(y,\lambda)$ we can compute the *net irradiance* $\overline{H}(y,\underline{n},\lambda)$ at y along any direction $\underline{n}$.

Another way of using (2.14) is to choose $\underline{n}$ in turn to be the cartesian coordinate axis directions $\underline{i}_1,\underline{i}_2,\underline{i}_3$ in Fig. 2. This in effect resolves $\underline{H}(y,\lambda)$ into its cartesian components. Thus write

$$'H_j(y,\lambda)' \quad \text{for} \quad \underline{i}_j \cdot \underline{H}(y,\lambda) \tag{2.15}$$
$$j = 1,2,3.$$

By (2.10), (2.11) and (2.13), (2.14), it follows that

$$\begin{aligned} H_j(y,\lambda) &= H(y,\underline{i}_j,\lambda) - H(y,-\underline{i}_j,\lambda) \ , \quad j = 1,2,3 \\ &= \overline{H}(y,\underline{i}_j,\lambda) \end{aligned}$$

and by vector algebra we have in general

$$\underline{H}(y,\lambda) = \underline{i}_1 H_1(y,\lambda) + \underline{i}_2 H_2(y,\lambda) + \underline{i}_3 H_3(y,\lambda) \ . \tag{2.16}$$

As a vector, $\underline{H}(y,\lambda)$ has a direction given by the unit vector

$$\underline{m} \equiv \underline{H}(y,\lambda)/|\underline{H}(y,\lambda)| \tag{2.17a}$$

where

$$|\underline{H}(y,\lambda)| \equiv |\underline{H}(y,\lambda)| = [H_1^2(y,\lambda) + H_2^2(y,\lambda) + H_3^2(y,\lambda)]^{\frac{1}{2}} \ . \tag{2.17b}$$

Then we can write $\underline{H}(y,\lambda)$ as

$$\underline{H}(y,\lambda) = \underline{m}|\underline{H}(y,\lambda)| \tag{2.17c}$$

From (2.14) and (2.17c), we find that, for any direction $\underline{n}$:

$$\begin{aligned} \overline{H}(y,\underline{n},\lambda) &= \underline{n}\cdot\underline{H}(y,\lambda) = \underline{n}\cdot\underline{m}|\underline{H}(y,\lambda)| \\ &= \cos\psi|\underline{H}(y,\lambda)| \end{aligned} \tag{2.18}$$

where ψ is the angle between the arbitrary direction $\underline{n}$ and the fixed direction $\underline{m}$ of $\underline{H}(y,\lambda)$ (cf. Fig. 5).

By this we obtain the desired intuitive interpretation of $\underline{H}(y,\lambda)$: The direction $\underline{m}$ of $\underline{H}(y,\lambda)$ is that of the greatest net irradiance $\bar{H}(y,\underline{m},\lambda)$ at y, and the length $|\underline{H}(y,\lambda|$ is equal to this maximum net irradiance.

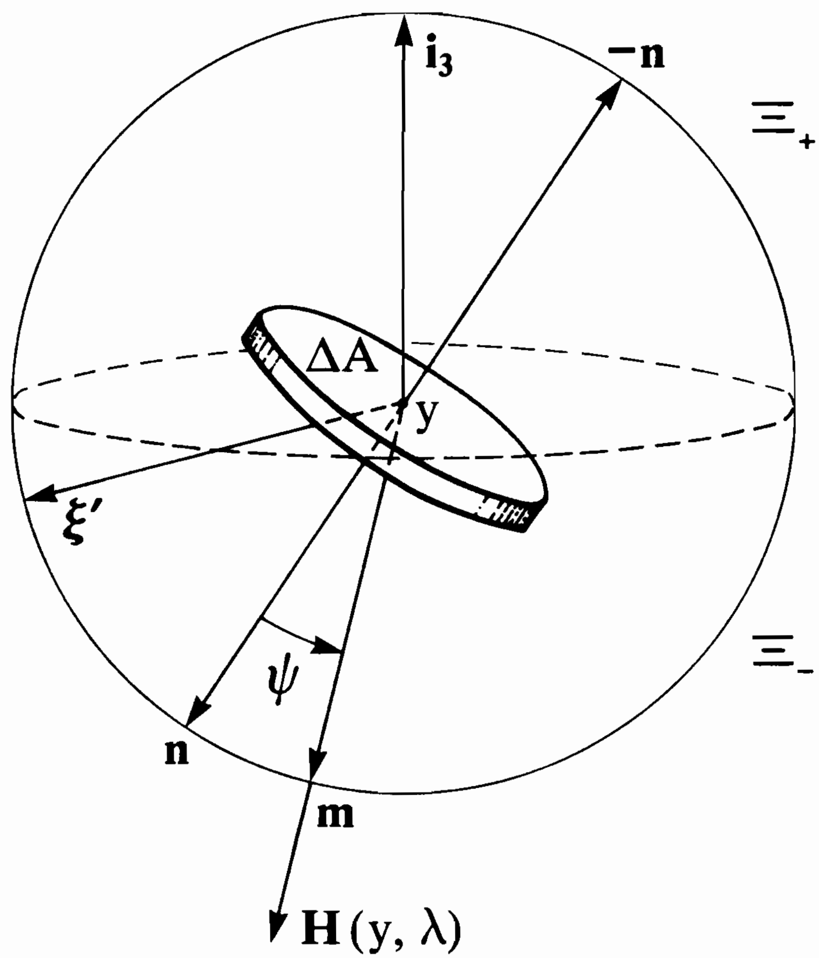

Figure 5--Geometry used in the definition of the vector irradiance $\underline{H}(y,\lambda)$.

3. VOLUME SCATTERING FUNCTION, MONOCHROMATIC CASE

The most important optical property of a scattering-absorbing optical medium in which radiative transfer of energy takes place is the volume scattering function. In the present section we define this concept for the case of elastic scattering--i.e., scattering without change in wavelength, the *monochromatic case*. In the next section the heterochromatic case is considered.

The *in situ* experimental setup for photon scattering is shown schematically in plan view by Fig. 6. A monochromator-collimator device shoots a narrow beam of radiant flux of wavelength λ from point A to point y, and beyond in the scattering absorbing medium. All along the extent of the beam photons are being absorbed and scattered by the molecules of the medium. The scattering activity is examined in detail at point y of the beam by a radiance meter. The narrow beam from A to y and the narrow field of view of y as seen at B define an element of volume R in the medium about y, as shown in the inset of Fig. 6. This plan view of the volume is in effect an optically small parallelogram of dimensions depicted in the inset. The photons in R are scattered throughout the volume, and those that go off toward B produce an observed spectral radiance $N^*_{\Delta r}(y,\underline{\xi},\lambda)$ at y, along $\underline{\xi}$, of wavelength λ, in accordance with (2.1). This radiance is generated within the short path of length Δr (in m) in R along the direction $\underline{\xi}$. Thus the spectral radiance generated per unit length along $\underline{\xi}$ is $N^*_{\Delta r}(y,\underline{\xi},\lambda)/\Delta r$. The photons of wavelength λ' (= λ) from A giving rise to this scattered radiance are generated by unpolarized, incoherent incident radiant flux of spectral irradiance $N(y,\underline{\xi}',\lambda)\Delta\Omega'$ over the normally projected incident face of R, normal to $\underline{\xi}'$, and of area $\Delta A'$. We then expect $N^*_{\Delta r}(y,\underline{\xi},\lambda)/\Delta r$, for fixed Δr, y, $\underline{\xi}$, $\underline{\xi}'$ and λ to vary in direct proportion to this incident irradiance $N(y,\underline{\xi}',\lambda)\Delta\Omega'$.

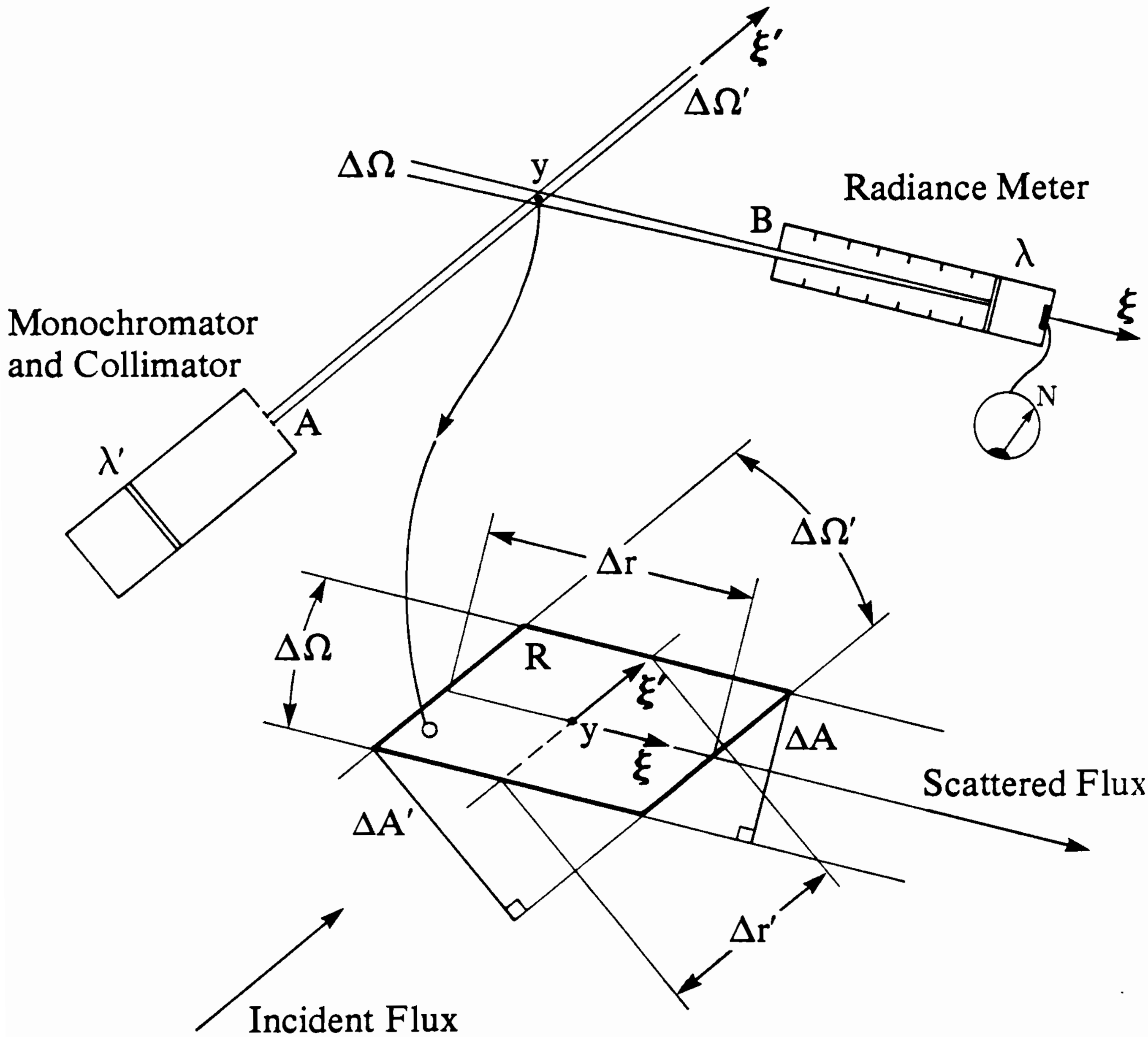

Figure 6--The setup used in the experimental determination of the volume scattering function $\sigma(y;\underline{\xi}';\underline{\xi};\lambda)$.

The proportionality constant is by definition the value of the *volume scattering function* $\sigma(y;\underline{\xi}';\underline{\xi};\lambda)$, where we write

$$'\sigma(y;\underline{\xi}';\underline{\xi};\lambda)' \quad \text{for} \quad \frac{N^*_{\Delta r}(y,\underline{\xi},\lambda)/\Delta r}{N(y,\underline{\xi}',\lambda)\Delta\Omega'} \qquad (\mathrm{m}^{-1}\cdot\mathrm{sr}^{-1}). \tag{3.1}$$

Let $P^*_{\Delta V}(y,\underline{\xi},\lambda)$ be the spectral radiant flux (in $\mathrm{W}\cdot\mathrm{nm}^{-1}$, cf. (2.1)) generated by the scattering region R of volume $\Delta V = \Delta r\Delta A$, and set $\Delta H(y,\underline{\xi}',\lambda) = N(y,\underline{\xi}',\lambda)\Delta\Omega'$. From simple geometric considerations of the parallelepiped R in Fig. 6, and on observing via (2.1) that $N^*_{\Delta r}(y,\underline{\xi},\lambda) \equiv P^*_{\Delta V}(y,\underline{\xi},\lambda)/\Delta A\Delta\Omega$, we may also write (3.1) as

$$\begin{aligned}\sigma(y;\underline{\xi}';\underline{\xi};\lambda) &= \frac{P^*_{\Delta V}(y,\underline{\xi},\lambda)/\Delta\Omega}{\Delta V}\cdot\frac{1}{\Delta H(y,\underline{\xi}',\lambda)}\\ &\equiv \frac{J^*_{\Delta V}(y,\underline{\xi},\lambda)}{\Delta V}\cdot\frac{1}{\Delta H(y,\underline{\xi}',\lambda)}\end{aligned} \tag{3.2}$$

This alternate view of σ allows the role of the volume ΔV of R to be discerned. $J^*_{\Delta V}(y,\underline{\xi},\lambda) \equiv P^*_{\Delta V}(y,\underline{\xi},\lambda)/\Delta\Omega$ is the *spectral radiant intensity* (in $\mathrm{W}\cdot\mathrm{sr}^{-1}\cdot\mathrm{nm}^{-1}$) of this volume produced by the scattered photons in R.

The form of σ most appropriate for theoretical and practical radiometry is given in (3.1), which we rewrite as

$$\sigma(y;\underline{\xi}';\underline{\xi};\lambda) \equiv \frac{N_*(y,\underline{\xi},\lambda)}{N(\underline{y},\underline{\xi}',\lambda)\Delta\Omega'} \qquad (\mathrm{m}^{-1}\cdot\mathrm{sr}^{-1}) \tag{3.3}$$

where $N_*(y,\underline{\xi},\lambda)$ is the *spectral* (monochromatic) *path function*:

$$N_*(y,\underline{\xi},\lambda) \equiv N^*_{\Delta r}(y,\underline{\xi},\lambda)/\Delta r \qquad (\mathrm{W}\cdot\mathrm{m}^{-3}\cdot\mathrm{sr}^{-1}\cdot\mathrm{nm}^{-1}) \tag{3.4}$$

The form (3.3) is useful because it converts immediately into an integral representation of the path function:

$$N_*(y,\underline{\xi},\lambda) \equiv \int_{\Xi} N(y,\underline{\xi}',\lambda)\ \sigma(y;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}') \qquad (W\cdot m^{-3}\cdot sr^{-1}\cdot nm^{-1}) \tag{3.5}$$

which forms part of the equation of transfer, derived below. The relation (3.5) shows how the radiance per unit length $N_*(y,\underline{\xi},\lambda)$ at y along $\underline{\xi}$ is generated by the scattering into the direction $\underline{\xi}$ of photons arriving at y along all directions $\underline{\xi}'$ in the unit sphere Ξ.

The reverse of the operation in (3.5) is to sum up the radiant flux scattered out of R in all directions $\underline{\xi}$ about y. The result is denoted in general by writing

$$\text{'}s(y,\underline{\xi}',\lambda)\text{'} \quad \text{for} \quad \int_{\Xi} \sigma(y;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}) \qquad (m^{-1}) \tag{3.6}$$

The quantity $s(y,\underline{\xi}',\lambda)$ is the *volume total scattering function* at y. In most natural hydrosols $s(y,\underline{\xi}',\lambda)$ is independent of $\underline{\xi}'$ and we shall henceforth write '$s(y,\lambda)$' for $s(y,\underline{\xi}',\lambda)$.

In analogy to the scalar irradiance $h(y,\lambda)$ in (2.9b), the integral of the path function over Ξ generates the *scattered scalar irradiance* $h_*(y,\lambda)$, where we write

$$\text{'}h_*(y,\lambda)\text{'} \quad \text{for} \quad \int_{\Xi} N_*(y,\underline{\xi},\lambda)\ d\Omega(\underline{\xi}) \qquad (W\cdot m^{-3}\cdot nm^{-1}) \tag{3.7}$$

Using (3.5) and (3.6) the scattered scalar irradiance has the representation

$$h_*(y,\lambda) = \int_\Xi \left[\int_\Xi N(y,\underline{\xi}',\lambda)\ \sigma(y;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}') \right] d\Omega(\underline{\xi})$$

$$= \int_\Xi N(y,\underline{\xi}',\lambda) \left[\int_\Xi \sigma(y;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}) \right] d\Omega(\underline{\xi}')$$

$$= s(y,\lambda) \int_\Xi N(y,\underline{\xi}',\lambda) d\Omega(\underline{\xi}')$$

i.e.,

$$h_*(y,\lambda) = s(y,\lambda)\ h(y,\lambda)\ . \tag{3.8}$$

This relation describes the scattering losses of radiant flux at point y in all directions, from a unit volume about y, given the scalar irradiance $h(y,\lambda)$ at y.

A sharper relation of the kind shown in (3.8) comes from (3.5) in which the incident radiance $N(y,\underline{\xi}',\lambda)$ is confined to a single narrow beam along direction $\underline{\xi}_o$. Thus, using the dirac delta function, δ, on Ξ (so that the dimensions of δ are sr^{-1}) we set

$$N(y,\underline{\xi}',\lambda) \equiv h'(y,\underline{\xi}',\lambda)\ \delta(\underline{\xi}_o - \underline{\xi}') \tag{3.9}$$

Here $h'(y,\underline{\xi}',\lambda)$ (in $W \cdot m^{-2} \cdot nm^{-1}$) is the *scalar irradiance of the beam.* Then (3.5) reduces to

$$N_*(y,\underline{\xi},\lambda) = h'(y,\underline{\xi}_o,\lambda)\ \sigma(y;\underline{\xi}_o;\underline{\xi};\lambda) \tag{3.10}$$

and the scattered radiant flux loss from the unit volume at y irradiated by this beam along $\underline{\xi}_o$ is

$$h_*(y,\lambda) = s(y,\lambda)\ h'(y,\underline{\xi}_o,\lambda) \tag{3.11}$$

4. VOLUME SCATTERING FUNCTION, HETEROCHROMATIC CASE

We return to the setting depicted in Fig. 6 and now irradiate the element of volume R by radiant flux of wavelength λ' over a band $\Delta\lambda'$ distinct from the wavelength λ for which the radiance meter is set. In some materials, such as various compounds comprising chlorophyll molecules, the incident radiant flux $\Delta H(y,\underline{\xi}',\lambda')\Delta\lambda'$ on R excites an emission of spectral radiance $N^{*}_{\Delta r}(y,\underline{\xi},\lambda)$ at this distinct and usually longer wavelength λ. In analogy with (3.1) we then write, for $\lambda' \neq \lambda$,

$$'\hat{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)' \quad \text{for} \quad \frac{N^{*}_{\Delta r}(y,\underline{\xi},\lambda)/\Delta r}{N(y,\underline{\xi}',\lambda')\Delta\Omega'\Delta\lambda'} \qquad (m^{-1}\cdot sr^{-1}\cdot nm^{-1}) \quad (4.1)$$

Also, in analogy with (3.4), for small r, we write, in the present heterochromatic case,

$$'N_s(y,\underline{\xi},\lambda)' \quad \text{for} \quad N^{*}_{\Delta r}(y,\underline{\xi},\lambda)/\Delta r \qquad (W\cdot m^{-3}\cdot sr^{-1}\cdot nm^{-1}) \qquad (4.2)$$

which is the *transpectral path function* (Preisendorfer, 1965, p. 59).

We can now define the heterochromatic counterpart to (3.5) by writing

$$'N_s(y,\underline{\xi},\lambda)' \quad \text{for} \quad \int_{\Xi} \int_{\Lambda} N(y,\underline{\xi}',\lambda')\, \hat{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)d\lambda' d\Omega(\underline{\xi}')$$

$$(W\cdot m^{-3}\cdot sr^{-1}\cdot nm^{-1}) \qquad (4.3)$$

where Λ is the electromagnetic spectrum $[0,\infty]$ or some subset of Λ such as [400 nm, 700 nm]. Eq. (4.3) describes the space rate of gain at y along the direction $\underline{\xi}$ of radiance generated by directional and transpectral scattering of radiant flux. The integration over Λ in (4.3) is defined even though $\hat{\sigma}$ has been constructed only for $\lambda' \neq \lambda$. The value of $\hat{\sigma}$ in singular case $\lambda' = \lambda$ may be arbitrarily assigned and the result (by a basic property of Riemann integrals) will not contribute to the integral of $\hat{\sigma}$ over Λ. Accordingly, we shall set the value of $\hat{\sigma}$ for $\lambda' = \lambda$ to zero: $\hat{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda,\lambda) = 0$ for all λ in Λ.

The *volume total transpectral scattering function* is defined by writing

$$'\hat{s}(y;\underline{\xi}';\lambda',\lambda)' \quad \text{for} \quad \int_{\Xi} \hat{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)\, d\Omega(\underline{\xi}) \qquad (m^{-1}\cdot nm^{-1}) \qquad (4.4)$$

As in the monochromatic case (cf. (3.6)) the function $\hat{s}(y;\underline{\xi}';\lambda',\lambda)$ is independent of the incident direction $\underline{\xi}'$ in most natural hydrosols. Therefore we will shorten '$\hat{s}(y;\underline{\xi}';\lambda',\lambda)$' to '$\hat{s}(y,\lambda',\lambda)$'.

In analogy to the scattered scalar irradiance $h_*(y,\lambda)$ in (3.8), we have the *transpectrally scattered scalar irradiance*:

$$h_s(y,\lambda) = \int_{\Lambda} h(y,\lambda')\, \hat{s}(y,\lambda',\lambda)\, d\lambda' \qquad (4.5)$$

which follows from (4.3), (4.4), and the definition

$$'h_s(y,\lambda)' \quad \text{for} \quad \int_{\Xi} N_s(y,\underline{\xi},\lambda)\, d\Omega(\underline{\xi}) \qquad (W\cdot m^{-3}\cdot nm^{-1}) \qquad (4.6)$$

5. VOLUME ABSORPTION FUNCTIONS: TRUE, TRANSPECTRAL, AND TOTAL

The introduction of the volume absorption function into radiative transfer theory requires care in accounting for the particular fate of an absorbed photon (cf. Preisendorfer, 1965, p. 60). There are essentially two distinct cases of absorption-type activity to consider when a photon of wavelength λ' in the incident beam of Fig. 6 arrives in region R about point y: In case (i), the photon is truly absorbed by an electron, and the energy of the photon is converted into non-radiant energy [potential (ionized, orbital) energy of the capturing electron, kinetic energy of the recoiling molecule, potential energy required to liberate an oxygen molecule, etc.]. This action is called *true absorption*. In case (ii), the photon's energy is used by a molecule to momentarily raise the electron's or molecule's orbital energy, and then the energy is subsequently released (as described by $\hat{\sigma}$ in (4.1)) in the form of a photon of a generally (but not always) longer wavelength λ. This action is called *transpectral absorption* because, as far as the incident radiant flux of that wavelength λ is concerned, there has been a loss or an 'absorption' from flux of *that wavelength*.

We shall denote the *true volume absorption function* by '$a_e(y,\lambda')$' (in m^{-1}) and the *transpectral volume absorption function* is, in accordance with the above description, defined by writing

$$'\hat{a}(y,\lambda')' \quad \text{for} \quad \int_{\Lambda} \hat{s}(y,\lambda',\lambda)\, d\lambda \qquad (m^{-1}) \tag{5.1}$$

Both a_e and $\hat{a}$ are to be used in mathematical statements in the same way that the volume total scattering function s is used. In more detail, and being guided by (3.11) and Fig. 6, suppose a beam of photons of radiance $N(y,\underline{\xi}',\lambda)$ impinges on region R and undergoes losses only by scattering of photons. After passage a distance $\Delta r'$ through R, let the beam have radiance $N(y+\Delta r'\underline{\xi}',\underline{\xi}',\lambda')$. By the physical meaning of (3.11), the latter radiance is given approximately by

$$N(y+\Delta r'\underline{\xi}',\xi',\lambda') = N(y,\underline{\xi}',\lambda') - s(y,\lambda')\ \Delta r'\ N(y,\underline{\xi}',\lambda') \tag{5.2}$$

We want $a_e(y,\lambda')$ and $\hat{a}(y,\lambda')$ to account for absorptive losses to radiance in a similar way. Thus, for the two absorptive cases described above, and excluding losses by scattering we write

$$N(y+\Delta r',\underline{\xi}',\lambda') = N(y,\underline{\xi}',\lambda') - [\hat{a}(y,\lambda') + a_e(y,\lambda')]\Delta r'\ N(y,\underline{\xi}',\lambda') \tag{5.3}$$

The total space rate of loss of $N(y,\underline{\xi}',\lambda')$ at y by absorption and scattering is then generally given by

$$\frac{dN(y,\underline{\xi}',\lambda')}{dr'} = -[a(y,\lambda') + s(y,\lambda')]\ N(y,\underline{\xi}',\lambda') \tag{5.4}$$

where we have written

$$'a(y,\lambda')' \quad \text{for} \quad \hat{a}(y,\lambda') + a_e(y,\lambda') \qquad (m^{-1}) \tag{5.5}$$

We call $a(y,\lambda')$ the *volume total absorption* function evaluated at y for wavelength λ'. In sum then, $a(y,\lambda')$ consists of two types of absorption: transpectral absorption $\hat{a}$ (loss by scattering to other wavelengths) and true absorption or *extinction* absorption a_e (conversion to non-radiant energy). The 'e' in a_e serves to remind one of the extinction of radiant energy in this sense.

6. FLUORESCENCE FUNCTION, EFFICIENCY FUNCTION

The volume total transpectral scattering function $\hat{s}(y,\lambda',\lambda)$ defined in (4.4) can be split into three factors, each of basic physical importance in the theory of the fluorescent light field to be developed below.

To begin, we examine the λ dependence of $\hat{s}(y;\lambda',\lambda)$ for fixed incident wavelength λ'. Let us define the *fluorescence function* by writing

$$'\phi(y,\lambda',\lambda)' \quad \text{for} \quad \hat{s}(y,\lambda',\lambda)/\hat{a}(y,\lambda') \qquad (\text{nm}^{-1}) \tag{6.1}$$

where $\hat{a}(y,\lambda')$ is defined in (5.1). Hence ϕ is a normalized function of λ in the sense that

$$\int_{\Lambda} \phi(y,\lambda',\lambda)d\lambda = 1 \ . \tag{6.2}$$

By means of experimental observations of fluorescing material (Pringsheim, 1963), it has been found that ϕ's λ-behavior is nearly Gaussian, and largely independent of λ'. Thus we can represent $\phi(y;\lambda',\lambda)$ by

$$\phi(y,\lambda',\lambda) = [2\pi w^2]^{-\frac{1}{2}} \exp\{-(\lambda-\lambda_o)^2/2w^2\} \tag{6.3}$$

Here λ_o is the wavelength about which the fluorescence peaks, and w is the 'half width' of the peak. Another possibility for ϕ, particularly suited for isolated absorption lines, is the Cauchy function

$$\phi(y,\lambda',\lambda) = (2/\pi)\ \gamma/[(\lambda-\lambda_o)^2 + \gamma^2] \tag{6.4}$$

Generally any fluorescing material's fluorescence function can be represented through least square fits of suitable linear combinations of the Gaussian functions in (6.3) or Cauchy functions in (6.4).

From (6.1) we have the representation of $\hat{s}$,

$$\hat{s}(y,\lambda',\lambda) = \hat{a}(y,\lambda')\ \phi(y,\lambda',\lambda) \tag{6.5}$$

We can introduce into (6.5) the total absorption function $a(y,\lambda)$ by recalling (5.5) and writing $\hat{a}(y,\lambda')$ as

$$\begin{aligned}\hat{a}(y,\lambda') &= a(y,\lambda') - a_e(y,\lambda') \\ &= a(y,\lambda')\left[1 - \frac{a_e(y,\lambda')}{a(y,\lambda')}\right]\end{aligned}$$

i.e.,

$$\hat{a}(y,\lambda') = a(y,\lambda')\ \eta(y,\lambda') \tag{6.6}$$

where we have written

$$\text{'}\eta(y,\lambda')\text{' for } 1 - \frac{a_e(y,\lambda')}{a(y,\lambda')} \tag{6.7}$$

$$= \hat{a}(y,\lambda')/a(y,\lambda') \tag{6.8}$$

$\eta(y,\lambda')$ is the *fluorescence efficiency* of the medium at y for incident wavelength λ'. Clearly, if $a_e(y,\lambda') = 0$, then $\eta(y,\lambda') = 1$ and all the absorbed radiant energy has been transpectrally scattered, i.e., has fluoresced. Hence 'efficiency' here refers to the efficiency of the material to fluoresce. With this definition we attain the desired set of representations of $\hat{s}$:

$$\hat{s}(y,\lambda',\lambda) = \hat{a}(y,\lambda')\ \phi(y,\lambda',\lambda) \tag{6.9}$$

$$= a(y,\lambda')\ \eta(y,\lambda')\ \phi(y,\lambda',\lambda) \tag{6.10}$$

$$= a(y,\lambda')\ \hat{\eta}(y,\lambda',\lambda) \tag{6.11}$$

where we have written

$$'\hat{\eta}(y,\lambda',\lambda)' \text{ for } \eta(y,\lambda')\ \phi(y,\lambda',\lambda) \tag{6.12}$$

and which we call the *transpectral efficiency* of the medium at y for the wavelength pair (λ',λ).

This last form of $\hat{s}$ will be useful later in drawing an analogy between geometric scattering (from $\underline{\xi}'$ to $\underline{\xi}$) and spectral scattering (from λ' to λ).

Finally, we observe that a quantum (or photon) version of the transpectral scatter $\hat{s}(y,\lambda',\lambda)$ and transpectral efficiency $\hat{\eta}(y,\lambda',\lambda)$ can be obtained by placing (4.5) into quantum form using the radiometric-to-quantum conversion rule (2.2). Thus, multiplying each side of (4.5) by $\lambda/h_o c$, and rearranging λ',λ, we find

$$[h_s(y,\lambda)\lambda/h_o c] = \int_\Lambda [h(y,\lambda')\lambda'/h_o c][\hat{s}(y,\lambda',\lambda)\lambda/\lambda']d\lambda' \tag{6.13}$$

From (6.11) we find the required quantum forms of $\hat{s}$ and $\hat{\eta}$:

$$\hat{s}(y,\lambda',\lambda)\lambda/\lambda' = a(y,\lambda')[\hat{\eta}(y,\lambda',\lambda)\lambda/\lambda'] \tag{6.14}$$

The units of $h_s(y,\lambda)\lambda/h_o c$ are photon$\cdot$s$^{-1}\cdot$m$^{-3}\cdot$nm^{-1}. Hence $\hat{s}\lambda/\lambda'$ and $\hat{\eta}\lambda/\lambda'$ refer to the transpectral scatter of *quanta* of the light field from λ' to λ.

7. COMPLETE VOLUME SCATTERING FUNCTION; VOLUME ATTENUATION FUNCTION

The two volume scattering functions σ and $\hat{\sigma}$ of sections 3 and 4, respectively, can be combined into a single *complete volume scattering function*:

$$\tilde{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda) \equiv \sigma(y;\underline{\xi}';\underline{\xi};\lambda')\ \delta(\lambda-\lambda') + \hat{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda) \quad (\mathrm{m}^{-1}\cdot\mathrm{sr}^{-1}\cdot\mathrm{nm}^{-1}) \tag{7.1}$$

where $\delta(\lambda-\lambda')$ (in nm^{-1}) is the dirac delta function on Λ.
In analogy to the fluorescence function $\phi(y;\lambda',\lambda)$ we can define the *monochromatic* and *heterochromatic phase functions*:

$$'p(y;\underline{\xi}';\underline{\xi};\lambda)' \text{ for } \sigma(y;\underline{\xi}';\underline{\xi};\lambda)/s(y,\lambda) \qquad (\mathrm{sr}^{-1}) \tag{7.2}$$

$$'\hat{p}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)' \text{ for } \hat{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)/\hat{s}(y,\lambda',\lambda) \qquad (\mathrm{sr}^{-1}) \tag{7.3}$$

so that p and $\hat{p}$ are normalized in the sense that

$$\int_{\Xi} p(y;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}) = 1 \tag{7.4}$$

and

$$\int_{\Xi} \hat{p}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)\ d\Omega(\underline{\xi}) = 1 \tag{7.5}$$

Hence we can represent σ and $\hat{\sigma}$ as

$$\sigma(y;\underline{\xi}';\underline{\xi};\lambda) = s(y,\lambda')\ p(y;\underline{\xi}';\underline{\xi};\lambda') \tag{7.6}$$

$$\begin{aligned}\hat{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda) &= a(y,\lambda')\ \eta(y,\lambda')\ \phi(y;\lambda',\lambda)\ \hat{p}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda) \\ &= \hat{s}(y,\lambda',\lambda)\ \hat{p}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)\end{aligned} \tag{7.7}$$

The representations in (7.6), (7.7) draw out some analogies between 'transpectral' scatter and 'transdirectional' scatter on comparing the λ',λ behavior of the normalized functions $\phi(y;\lambda',\lambda)$ and the $\underline{\xi}',\underline{\xi}$ behavior of $p(y;\underline{\xi}';\underline{\xi};\lambda)$. Also $s(y,\lambda')$ and $\hat{s}(y,\lambda',\lambda)$ are corresponding total-type losses in (7.6), (7.7). Finally, it is clear that the transpectral efficiency $\eta(y,\lambda')$ is the phenomenological link between absorption and heterochromatic scattering, as brought out in (6.11). In summary, in (7.1), via (7.6) and (7.7), s and $\hat{s}$ carry the locational scattering behavior (in units m^{-1}); p and $\hat{p}$ carry the directional behavior (in units sr^{-1}); and $\delta(\lambda-\lambda')$ (via (7.1)) and $\phi(y;\lambda',\lambda)$ carry the spectral behavior (in units nm^{-1}) of $\tilde{\sigma}$.

The total loss of photons from a beam under steady state conditions can now be completely accounted for on the phenomenological level. By integrating (7.1) over Ξ we find

$$\tilde{s}(y;\lambda',\lambda) \equiv \int_{\Xi} \tilde{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)\ d\Omega(\underline{\xi}) = \hat{s}(y,\lambda',\lambda) + s(y,\lambda')\ \delta(\lambda-\lambda') \tag{7.8}$$

Further, on integrating $\tilde{s}(y,\lambda',\lambda)$ over λ we find (on recalling (5.1)):

$$\int_{\Lambda} \tilde{s}(y,\lambda',\lambda)\ d\lambda = \hat{a}(y,\lambda') + s(y,\lambda') \tag{7.9}$$

On adding $a_e(y,\lambda')$ to each side of (7.9) we find

$$\int_{\Lambda} \tilde{s}(y,\lambda',\lambda)\, d\lambda + a_e(y,\lambda) = \hat{a}(y,\lambda') + a_e(y,\lambda') + s(y,\lambda')$$

$$= a(y,\lambda') + s(y,\lambda')$$

$$\equiv \alpha(y,\lambda') \qquad (m^{-1}) \qquad (7.10)$$

where $\alpha(y,\lambda')$ is the *volume attenuation function* at y and λ'. Thus $\alpha(y,\lambda')$ combines all three possible types of losses when a packet of photons, each of wavelength λ', impinges on R at y, as depicted in Fig. 6.

8. RADIANCE TRANSFER EQUATION

We may now assemble all the pieces, gathered so far, into the equation that describes the net spatial rate of gain of radiance along a path in an optical medium. We shall work in a plane-parallel medium X(a,b), as depicted in Fig. 7. Only portions of the planes are shown. These extend indefinitely in all horizontal directions, i.e., directions $\underline{\xi}$ normal to the upward direction $\underline{k}$. The medium X(a,b) is generally bounded by reflecting-transmitting media X(a,x), X(z,b) that can either be finite-thickness scattering-absorbing media, or the infinitesimally thin air-water surface and the bottom-surface of a natural hydrosol. The medium's optical properties are assumed constant on all planes parallel to the upper horizontal boundary plane at level a. Depth y is measured positive downward (in m).

Consider an unpolarized small packet of photons of wavelength λ and radiance $N(y,\underline{\xi},\lambda)$ at depth y streaming along direction $\underline{\xi}$ in Ξ. As the packet of photons travels a positive distance Δr (in m) along $\underline{\xi}$, it changes depth by an amount $-\Delta r\ \underline{k}\cdot\underline{\xi} \equiv \Delta y$. From (5.4) and (7.10) we can reckon the spatial rate of loss of $N(y,\underline{\xi},\lambda)$ by the processes of absorption and scattering as the packet moves along $\underline{\xi}$ at y. Equations (3.5) and (4.3) together give the space rate of gain of $N(y,\underline{\xi},\lambda)$ by monochromatic and heterochromatic scattering of photons into the direction $\underline{\xi}$ and into wavelength λ. Finally, let '$N_e(y,\underline{\xi},\lambda)$' (in $W\cdot m^{-3}\cdot sr^{-1}\cdot nm^{-1}$) denote the radiance gained at y per unit length along $\underline{\xi}$ by photons of wavelength λ created from non-radiant energy* (heat-producing turbulence, phosphorescence, bioluminescence, artificial light sources; or generally, any *emission* process, so 'e' stands for *emission*). Then on assembling these various rates, we have

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* If (8.1) below is to describe monochromatic radiative transfer only, set $\hat{\sigma}$ to zero. Then N_e can formally include the possibility of fluorescence.

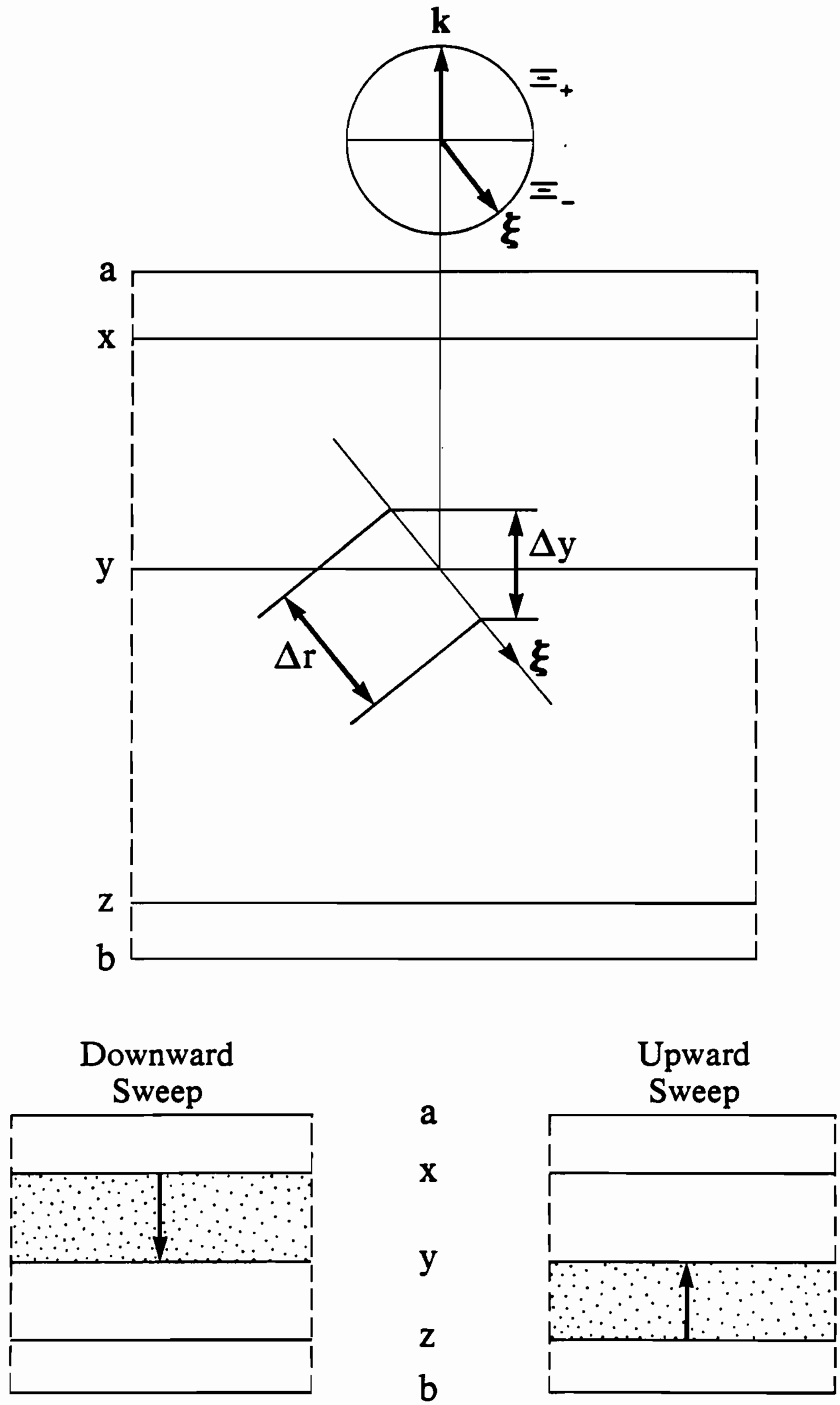

Figure 7--Geometry of the plane-parallel medium (top); the notation used for the upward and downward integration sweeps (bottom).

$$\boxed{\begin{aligned}-\underline{\xi}\cdot\underline{k}\,\frac{d}{dy}\,N(y,\underline{\xi},\lambda) = &- \alpha(y,\lambda)\;N(y,\underline{\xi},\lambda)\\ &+ \int_{\Xi}\int_{\Lambda} N(y,\underline{\xi}',\lambda')\;\tilde{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)\;d\lambda'd\Omega(\underline{\xi}')\\ &+ N_e(y,\underline{\xi},\lambda)\end{aligned}} \tag{8.1}$$

where α is given in (7.10) and $\tilde{\sigma}$ is given in (7.1). This is the required *radiance transfer* equation for fluorescing media.

A complete mathematical specification of the radiative transfer problem in the medium X(a,b) is made by adjoining to (8.1) the following information:

(i) The values of $\alpha(y,\lambda)$, $\tilde{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)$, and $N_e(y,\underline{\xi},\lambda)$ for all y in [x,y], $\underline{\xi}'$ and $\underline{\xi}$ in Ξ, and λ' and λ in Λ.

(ii) The four radiance reflectance and transmittance operators for each of the two boundary media X(a,x) and X(z,b) depicted in Fig. 7.

We now develop these reflectance and transmittance concepts and assemble them into boundary conditions on X(a,b) for (8.1). Generally, downward rays of light of radiance $N(a,\underline{\xi}',\lambda')$ uniformly incident over the lateral extent of X(a,x) along the direction $\underline{\xi}'$ in Ξ_-, and of wavelength λ' in $\Delta\lambda'$ undergo transpectral reflection and transmission. Let $\hat{r}(a,x;\underline{\xi}';\underline{\xi},\lambda',\lambda)$ (in $sr^{-1}\cdot nm^{-1}$) be the resultant relative amount of radiance reflected along $\underline{\xi}$ in Ξ_+ and of wavelength λ in $\Delta\lambda$. Then the spectral reflected radiance distribution at each point of the upper surface of X(a,x) is

$$N(a,\underline{\xi},\lambda) = \int_{\Xi_-}\int_{\Lambda} N(a,\underline{\xi}',\lambda')\;\hat{r}(a,x;\underline{\xi}';\underline{\xi};\lambda',\lambda)\;d\lambda'd\Omega(\underline{\xi}') \tag{8.2}$$

where $\underline{\xi}$ is in Ξ_+ (abbreviated as '$\underline{\xi}\ \varepsilon\ \Xi_+$'). Thus the use of $\hat{r}$ in (8.2) is basically similar to that of $\hat{\sigma}$ in (4.3), except that $\hat{r}$ pertains not to an element of volume but to a finite-thickness plane-parallel layer or a zero-thickness plane surface. In the present study we will restrict work to the zero-thickness air-water *surface* and a simple reflecting bottom *surface* at level b. However, to retain the symmetry and generality of the equations we adopt (as in a laboratory setup) a reflecting-transmitting (translucent) lower boundary. For such upper surfaces or bottom boundary layers, transpectral scattering is chosen (on physical grounds) not to be operative*. Thus we will restrict $\hat{r}$ to be the product $r(a,x;\underline{\xi}';\underline{\xi};\lambda)\ \delta(\lambda-\lambda')$, which is zero for $\lambda' \neq \lambda$. A corresponding form holds for the other seven boundary transfer functions defined below. The units of this *monochromatic* reflectance $r(a,x;\underline{\xi}';\underline{\xi};\lambda)$ are then sr^{-1}. In this way we arrive at the downward boundary reflectance operator $r(a,x)$.

The radiance $N(a,\underline{\xi},\lambda)$ leaving the infinitesimally thin surface $X(a,x)$ at level a in an upward direction is generally produced by both reflection (as shown above) and transmission of radiance from just below the surface. Therefore it is necessary to introduce also the notion of the monochromatic upward transmittance $t(x,a)$ operator of $X(a,x)$. The sum of these contributions is given by

$$N(a,\underline{\xi},\lambda) = \int_{\Xi_+} N(x,\underline{\xi}',\lambda)\ t(x,a;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}')$$

$$+ \int_{\Xi_-} N(a,\underline{\xi}',\lambda)\ r(a,x;\underline{\xi}';\underline{\xi},\lambda)\ d\Omega(\underline{\xi}')\ ,\quad \underline{\xi} \in \Xi_+ \qquad (8.3a)$$

and this constitutes the first of the four boundary conditions for (8.1). This boundary condition, as those in (8.2b,c,d) below are justified by the

* This constitutes no essential loss of generality of the present theory, as can be shown by use of the union rule developed in section 17.

linear interaction principle on which radiative transfer theory in geophysical applications is based (cf. H.O., vol. II). The units of t(x,a), as those of r(a,x), are sr^{-1}. The second condition for the upper boundary X(a,x) is

$$N(x,\underline{\xi},\lambda) = \int_{\Xi_-} N(a,\underline{\xi}',\lambda)\ t(a,x;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}') + \int_{\Xi_+} N(x,\underline{\xi}',\lambda)\ r(x,a;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}')\ ,\quad \underline{\xi}\ \varepsilon\ \Xi_- \tag{8.2b}$$

The four functions r(a,x), t(x,a), r(x,a) and t(a,x) in (8.2a,b) may be computed for each wind speed using the procedure in Preisendorfer and Mobley (1985, 1986).

The remaining two boundary conditions for a translucent lower boundary are

$$N(b,\underline{\xi},\lambda) = \int_{\Xi_-} N(z,\underline{\xi}',\lambda)\ t(z,b;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}') + \int_{\Xi_+} N(b,\underline{\xi}',\lambda)\ r(b,z;\underline{\xi}';\underline{\xi},\lambda)\ d\Omega(\underline{\xi}')\ ,\quad \underline{\xi}\ \varepsilon\ \Xi_- \tag{8.2c}$$

and

$$N(z,\underline{\xi},\lambda) = \int_{\Xi_+} N(b,\underline{\xi}',\lambda)\ t(b,z;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}') + \int_{\Xi_-} N(z,\underline{\xi}',\lambda)\ r(z,b;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}')\ ,\quad \underline{\xi}\ \varepsilon\ \Xi_+ \tag{8.2d}$$

It should be noted that the plane surfaces at levels a and x in X(a,x) are taken to be separated by an infinitesimal distance. However we assign two 'depths' a and x = a+ (say) to the surface because experience working with

(8.2a-d) has shown that, writing the name of the water surface as '$X(a,x)$' helps maintain in a simple way the directional distinction needed, e.g., between $t(a,x)$ (the downward transmittance) and $t(x,a)$ (the upward transmittance) of the air-water surface. Further, this notation allows us to distinguish in (8.2a) between $N(a,\underset{\sim}{\xi},\lambda)$ incident from above and $N(x,\underset{\sim}{\xi},\lambda)$ incident from below the surface. That is, the notation reminds us that the latter radiance is for upward flux ($\underset{\sim}{\xi} \in \Xi_+$), and the former is for downward flux ($\underset{\sim}{\xi} \in \Xi_-$) incident on their respective sides of the air-water interface. Similar notational conventions hold for the lower boundary $X(z,b)$. As the developments proceed, the usefulness of the r,t notation adopted in (8.2) will become evident. Occasionally we shall write 'r_+' and 'r_-' as short for $r(x,a)$ and $r(a,x)$, respectively, etc. Also the set of radiances $\{N(a,\underset{\sim}{\xi},\lambda): \ \underset{\sim}{\xi} \in \Xi_-\}$ will be abbreviated '$N_-(a,\lambda)$', etc.

We may now state the *direct mathematical problem* centered on (8.1) and (8.2):

<u>Given</u>: A plane-parallel optical medium $X(a,b)$, as in Fig. 7, with optical properties α, $\tilde{\sigma}$, and N_e continuously specified throughout the water body $X(x,z)$ and constant on planes parallel to that at level a, along with specifications of the four $r_\pm$ and $t_\pm$ functions on each of the boundaries $X(a,x)$, $X(z,b)$, and finally, the incident boundary spectral radiance distributions $N_-(a,\lambda)$, $N_+(b,\lambda)$.

<u>Required</u>: The internal radiance distribution $N(y,\lambda)$, $x \le y \le z$, throughout the water body $X(x,z)$ and the emergent boundary spectral radiance distributions $N_+(a,\lambda)$, $N_-(b,\lambda)$.

In the present study we shall formulate and solve the direct problem for a much simpler setting, that of the spectral *irradiance* field. This entails a reduction of (8.1) and (8.2) to the irradiance level of description. This will be done in sec. 10. We shall also formulate the *inverse mathematical problem* for 8.1 on the irradiance level and indicate a solution. This will be done in Part IV, below.

9. FLUX CONSERVATION EQUATION

We need a statement of the conservation of energy principle in the present transpectral scattering context. This is obtained from (8.1) by integrating each side over Ξ. We shall do this term by term.

The left side of (8.1) becomes, by application of (2.3) and (2.13) (with $\underline{k}$ instead of $\underline{n}$ or $\underline{i}_3$),

$$-\int_{\Xi} \underline{\xi}\cdot\underline{k}\, \frac{dN(y,\underline{\xi},\lambda)}{dy}\, d\Omega(\underline{\xi}) = -\left[\frac{dH(y,+,\lambda)}{dy} - \frac{dH(y,-,\lambda)}{dy}\right] \equiv -\frac{d\bar{H}(y,+,\lambda)}{dy} \quad (9.1)$$

$$= \nabla\cdot\underline{H}(y,\lambda) \qquad (W\cdot m^{-3}\cdot nm^{-1}) \quad (9.2)$$

The form (9.2) follows from the stratification assumption, i.e., that the vector irradiance $\underline{H}(y,\lambda)$ is constant over horizontal planes. The minus sign on the left in (8.1) arises because of our decision to measure y positive downward, while $\underline{k}$ points upward.

The first term on the right of (8.1) by (2.9b) becomes

$$\int_{\Xi} -\alpha(y,\lambda)\, N(y,\underline{\xi},\lambda)\, d\Omega(\underline{\xi}) = -\alpha(y,\lambda)\, h(y,\lambda) \quad (9.3)$$

$$= -[a(y,\lambda) + s(y,\lambda)]\, h(y,\lambda) \qquad (W\cdot m^{-3}\cdot nm^{-1})$$

The second term on the right of (8.1) by (3.8) and (4.5) becomes

$$s(y,\lambda)\, h(y,\lambda) + \int_{\Lambda} h(y,\lambda')\, \hat{s}(y,\lambda',\lambda)\, d\lambda' \qquad (W\cdot m^{-3}\cdot nm^{-1})$$

The final term on the right of (8.1) becomes by definition the *true emission (source density)* term

$$\int_{\Xi} N_e(y,\underline{\xi},\lambda)\, d\Omega(\underline{\xi}) \equiv h_e(y,\lambda) \qquad (W\cdot m^{-3}\cdot nm^{-1}) \quad (9.4)$$

Assembling these pieces we obtain the required result, the *flux conservation equation*:

$$\boxed{\begin{aligned} -\frac{d\bar{H}(y,+,\lambda)}{dy} = \nabla\cdot\underline{H}(y,\lambda) &= -[\hat{a}(y,\lambda) + a_e(y,\lambda)]\, h(y,\lambda) \\ &\quad + \int_{\Lambda} h(y,\lambda')\, \hat{s}(y,\lambda',\lambda)\, d\lambda' \\ x \le y \le z \qquad\qquad &\quad + h_e(y,\lambda) \\ \lambda \in \Lambda \qquad\qquad & \end{aligned}} \tag{9.5}$$

The left side of this equation defines the net outward flow (or loss) of spectral radiant energy of wavelength λ ('λ-flux') from a unit volume of the stratified optical medium at depth y. The right side shows the component parts of the flow: inward flow (or gain) to the volume by transpectral scattering of λ-flux and true absorption of λ-flux in the first term, outward flow of λ-flux generated by transpectral scattering in the second term and outward λ-flux by true emission in the third term.

10. IRRADIANCE TRANSFER EQUATIONS

In accordance with our closing remarks in section 8, we will now reduce the radiance transfer equation (8.1) to the irradiance level. This requires that (8.1) be integrated twice, once over Ξ_+ and once over Ξ_- and reduced by repeated applications of (2.3) and (2.9a), and the introduction of various new optical properties. We shall do the reductions of (8.1) term by term.

Integrating the left side of (8.1) over Ξ_+ or Ξ_- yields (on setting $\underline{i}_3 = \underline{k}$):

$$- \int_{\Xi_\pm} \underline{\xi} \cdot \underline{k} \frac{dN(y,\underline{\xi},\lambda)}{dy} \, d\Omega(\underline{\xi}) = \mp \frac{dH(y,\pm,\lambda)}{dy} \tag{10.1}$$

The α-term in (8.1) becomes (using (2.9a)):

$$\begin{aligned} - \int_{\Xi_\pm} \alpha(y,\lambda) \, N(y,\underline{\xi},\lambda) \, d\Omega(\underline{\xi}) &= -\alpha(y,\lambda) \, h(y,\pm,\lambda) \\ &= -\alpha(y,\lambda)[h(y,\pm,\lambda)/H(y,\pm,\lambda)] \, H(y,\pm,\lambda) \\ &= -\alpha(y,\pm,\lambda) \, H(y,\pm,\lambda) \end{aligned} \tag{10.2}$$

Here we have written

$$\text{'}\alpha(y,\pm,\lambda)\text{'} \quad \text{for} \quad \alpha(y,\lambda) \, D(y,\pm,\lambda) \qquad (m^{-1}) \tag{10.3a}$$

and for later reference we write in a similar vein

$$\text{'}a(y,\pm,\lambda)\text{'} \quad \text{for} \quad a(y,\lambda) \, D(y,\pm,\lambda) \qquad (m^{-1}) \tag{10.3b}$$

$$\text{'}s(y,\pm,\lambda)\text{'} \quad \text{for} \quad s(y,\lambda) \, D(y,\pm,\lambda) \qquad (m^{-1}) \tag{10.3c}$$

where we have written

$$\text{'}D(y,\pm,\lambda)\text{'} \quad \text{for} \quad h(y,\pm,\lambda)/H(y,\pm,\lambda) \tag{10.4}$$

We call $D(y,\pm,\lambda)$ the (dimensionless) *distribution function* for the upward (+) or downward (−) flow of photons at depth y and wavelength λ. In practice $D(y,\pm,\lambda)$ is a mildly varying quantity with depth y for each flow and wavelength and is therefore given the status of an apparent optical property (of course not of the medium, but of the light field; although it can be linked to the optical properties of the medium quite directly). The dimensionless factor $D(y,\pm,\lambda)$ in (10.3) is an average distance traveled by photons through a unit-depth layer of medium at level y, and serves to account for the directional effects of the light field on the *inherent optical property* $\alpha(y,\lambda)$ defined in (7.10).

The integral term in (8.1) consists of the sum $N_* + N_s$ of two parts, as indicated by (3.5), (4.3), and (7.1). Under integration over $\Xi_\pm$, the monochromatic part N_* of the sum becomes

$$\int_{\Xi_\pm} N_*(y,\underline{\xi},\lambda)\, d\Omega(\underline{\xi})$$

$$+ \int_{\Xi_\pm} \left[\int_{\Xi} N(y,\underline{\xi}',\lambda)\, \sigma(y;\underline{\xi}';\underline{\xi};\lambda)\, d\Omega(\underline{\xi}')\right] d\Omega(\underline{\xi})$$

$$= \int_{\Xi_\pm} \left[\left(\int_{\Xi_+} + \int_{\Xi_-}\right) N(y,\underline{\xi}',\lambda)\, \sigma(y;\underline{\xi}';\underline{\xi};\lambda)\, d\Omega(\underline{\xi}')\right] d\Omega(\underline{\xi}) \qquad (10.5)$$

For the outer integration over Ξ_+ we find

$$\int_{\Xi_+} N_*(y,\underline{\xi},\lambda)\, d\Omega(\underline{\xi}) = H(y,+,\lambda)\, f(y,+,\lambda) + H(y,-,\lambda)\, b(y,-,\lambda) \qquad (10.6)$$

and likewise for the outer integration over Ξ_- we find

$$\int_{\Xi_-} N_*(y,\underline{\xi},\lambda)\, d\Omega(\underline{\xi}) = H(y,-,\lambda)\, f(y,-,\lambda) + H(y,+,\lambda)\, b(y,+,\lambda) \tag{10.7}$$

where we have defined the *forward* and *backward scattering functions* by writing:

$$'f(y,\pm,\lambda)' \quad \text{for} \quad \frac{1}{H(y,\pm,\lambda)} \int_{\Xi_\pm} [\int_{\Xi_\pm} N(y,\underline{\xi}',\lambda)\, \sigma(y;\underline{\xi}';\underline{\xi};\lambda)\, d\Omega(\underline{\xi}')]\, d\Omega(\underline{\xi}) \quad (m^{-1}) \tag{10.8}$$

and

$$'b(y,\pm,\lambda)' \quad \text{for} \quad \frac{1}{H(y,\pm,\lambda)} \int_{\Xi_\mp} [\int_{\Xi_\pm} N(y,\underline{\xi}',\lambda)\, \sigma(y;\underline{\xi}';\underline{\xi};\lambda)\, d\Omega(\underline{\xi}')\, d\Omega(\underline{\xi}) \quad (m^{-1}) \tag{10.9}$$

Observe that by (7.10) and the preceding definitions (10.3), (10.4),

$$\begin{aligned} \alpha(y,\pm,\lambda) &= a(y,\pm,\lambda) + s(y,\pm,\lambda) \\ &= a(y,\pm,\lambda) + f(y,\pm,\lambda) + b(y,\pm,\lambda) \end{aligned} \tag{10.10}$$

The heterochromatic part N_s of $N_* + N_s$, on integration over $\Xi_\pm$, may be reduced in a similar way. We find

$$\begin{aligned} &\int_{\Xi_\pm} N_s(y,\underline{\xi},\lambda)\, d\Omega(\underline{\xi}) \\ &= \int_\Lambda \{H(y,\pm,\lambda')\, \hat{f}(y,\pm,\lambda',\lambda) + H(y,\mp,\lambda')\, \hat{b}(y,\mp,\lambda',\lambda)\}\, d\lambda' \end{aligned} \tag{10.11}$$

where we have written

$$'\hat{f}(y,\pm,\lambda',\lambda)' \quad \text{for} \quad \frac{1}{H(y,\pm,\lambda')} \int_{\Xi_\pm} \left[\int_{\Xi_\pm} N(y,\underline{\xi}',\lambda')\ \hat{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)\ d\Omega(\underline{\xi}')\right] d\Omega(\underline{\xi}) \qquad (10.12)$$

$(m^{-1}\cdot nm^{-1})$

$$'\hat{b}(y,\pm,\lambda',\lambda)' \quad \text{for} \quad \frac{1}{H(y,\pm,\lambda')} \int_{\Xi_\mp} \left[\int_{\Xi_\pm} N(y,\underline{\xi}',\lambda')\ \hat{\sigma}(y;\underline{\xi}';\underline{\xi};\lambda',\lambda)\ d\Omega(\underline{\xi}')\right] d\Omega(\underline{\xi}) \qquad (10.13)$$

$(m^{-1}\cdot nm^{-1})$

Finally, the true emission (source density) term in (8.1) becomes, by definition (cf. (9.4), (2.9a))

$$h_e(y,\pm,\lambda) \equiv \int_{\Xi_\pm} N_e(y,\underline{\xi},\lambda)\ d\Omega(\underline{\xi}) \qquad (W\cdot m^{-3}\cdot nm^{-1}) \qquad (10.14)$$

Assembling these results and using (10.10) we find the required *irradiance transfer equations*:

$$\boxed{\begin{aligned} \mp \frac{dH(y,\pm,\lambda)}{dy} &= -[a(y,\pm,\lambda) + b(y,\pm,\lambda)]\ H(y,\pm,\lambda) + b(y,\mp,\lambda)\ H(y,\mp,\lambda) \\ &\quad + \int_\Lambda \{\hat{f}(y,\pm,\lambda',\lambda)\ H(y,\pm,\lambda') + \hat{b}(y,\mp,\lambda',\lambda)\ H(y,\mp,\lambda')\}\ d\lambda' \\ &\quad + h_e(y,\pm,\lambda) \\ x &\le y \le z \\ \lambda &\in \Lambda \end{aligned}} \qquad (10.15)$$

where y is depth in meters.

These coupled integrodifferential equations, as descriptions of the fluorescent light field, are completed by giving their boundary conditions at the upper and lower surfaces. For these, we draw on the radiance boundary conditions (8.2a-d) and suitably reduce them to the irradiance level. For example (8.2a) is reduced by multiplying each side by $\underline{\xi}\cdot\underline{k}$, and integrating over Ξ_+. The left side becomes $H(a,+,\lambda)$ and the right side of (8.2a) is reduced in a manner analogous to (10.5) and (10.11). The resultant set of irradiance boundary conditions is

for $X(a,x)$:
$$\boxed{\begin{aligned} H(a,+,\lambda) &= H(x,+,\lambda)\ t(x,a,\lambda) + H(a,-,\lambda)\ r(a,x,\lambda) \\ H(x,-,\lambda) &= H(x,+,\lambda)\ r(x,a,\lambda) + H(a,-,\lambda)\ t(a,x,\lambda) \end{aligned}} \qquad (10.16a)$$

and

for $X(z,b)$:
$$\boxed{\begin{aligned} H(z,+,\lambda) &= H(b,+,\lambda)\ t(b,z,\lambda) + H(z,-,\lambda)\ r(z,b,\lambda) \\ H(b,-,\lambda) &= H(b,+,\lambda)\ r(b,z,\lambda) + H(z,-,\lambda)\ t(z,b,\lambda) \end{aligned}} \qquad (10.16b)$$

where we have written, for example, in (10.16a):

$$(t_+ =)\ 't(x,a,\lambda)'\ \text{for}\ \frac{1}{H(x,+,\lambda)} \int_{\Xi_+} \Big[\int_{\Xi_+} N(x,\underline{\xi}',\lambda)\ t(x,a;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}')\Big]\ |\underline{\xi}\cdot k|\ d\Omega(\underline{\xi})$$

and (10.17)

$$(r_- =)\ 'r(a,x,\lambda)'\ \text{for}\ \frac{1}{H(a,-,\lambda)} \int_{\Xi_+} \Big[\int_{\Xi_-} N(a,\underline{\xi}',\lambda)\ r(a,x;\underline{\xi}';\underline{\xi};\lambda)\ d\Omega(\underline{\xi}')\Big]\ |\underline{\xi}\cdot k|\ d\Omega(\underline{\xi})$$

These definitions supply two of the four transfer functions for the upper surface. The remaining two, $t(a,x,\lambda)$ (= t_-) and $r(x,a,\lambda)$ (= r_+) for $X(a,x)$, are obtained from the above formulas by interchanging + and - signs and x and a in the respective equations. Thus, for example, $t(a,x,\lambda)$ comes from $t(x,a,\lambda)$ by changing + to - along with changing x to a and a to x in the equation for $t(x,a,\lambda)$. The expressions for t_+ and r_- over the lower boundary $X(z,b)$ are obtained by making the replacements $a \to z$, $x \to b$ in (10.17). Sign and depth-variable reversals in the resultant lower boundary formulas yield the remaining pair t_-, r_+ for the lower boundary. Observe that $r_\pm$, $t_\pm$ for both boundaries are dimensionless.

In practice the $r_{\pm}$, $t_{\pm}$ quartet for each boundary X(a,x) or X(z,b) is given as a set of four numbers derived from independent theory or measurement. The air-water surface transfer quartet is found as described in Preisendorfer and Mobley (1985, 1986). The lower boundary quartet is usually built either directly or indirectly on a matte surface. By 'indirectly' we mean that X(z,b) itself could be a composite medium consisting of a water body X(z,b') and a matte bottom x(b',b).

Conservation of radiant energy requires for each λ that $r_+ + t_+ = 1$ and $r_- + t_- = 1$ at each translucent non absorbing and non fluorescing boundary. This is the case for the air-water surface. On the other hand, in natural waters, the bottom boundary is usually considered an opaque matte surface. Then we set $t_{\pm} = 0$, and $r_{\pm}$ are arbitrary numbers in the unit interval [0,1].

The forward and backward scattering functions for both the monochromatic and heterochromatic cases can be split into three useful factors each, as follows. For the monochromatic f and b functions in (10.8) and (10.9), we have

$$f(y,\pm,\lambda) = D(y,\pm,\lambda)\ s(y,\lambda)\ \varepsilon_f(y,\pm,\lambda) \tag{10.18}$$

$$b(y,\pm,\lambda) = D(y,\pm,\lambda)\ s(y,\lambda)\ \varepsilon_b(y,\pm,\lambda) \tag{10.19}$$

where $D(y,\pm,\lambda)$ are the distribution functions (cf. (10.4)), $s(y,\lambda)$ the volume total scattering function (cf. (3.6)), and where we have written

$$'\varepsilon_f(y,\pm,\lambda)' \quad \text{for} \quad \frac{\int_{\Xi_\pm} \left[\int_{\Xi_\pm} N(y,\xi',\lambda)\ p(y;\xi';\xi;\lambda)\ d\Omega(\xi')\right] d\Omega(\underline{\xi})}{\int_{\Xi_\pm} N(y,\underline{\xi}',\lambda)\ d\Omega(\underline{\xi}')} \tag{10.20}$$

$$'\varepsilon_b(y,\pm,\lambda)' \quad \text{for} \quad \frac{\int_{\Xi_\mp}\left[\int_{\Xi_\pm} N(y,\xi',\lambda)\, p(y;\xi';\xi;\lambda)\, d\Omega(\xi')\right] d\Omega(\underline{\xi})}{\int_{\Xi_\pm} N(y,\underline{\xi}',\lambda)\, d\Omega(\underline{\xi}')} \tag{10.21}$$

where $p(y;\underline{\xi}';\underline{\xi};\lambda)$ is the monochromatic phase function defined in (7.2).

The factors $D(y,\pm,\lambda)$, as already observed, are indexes of the shape of the radiance distributions in the $\Xi_\pm$ hemispheres, respectively. $s(y,\lambda)$ gives the total scattering activity of the medium at depth y. Finally, ε_f and ε_b are the (dimensionless) *eccentricities* for the monochromatic *forward* (f) and *backward* (b) scattering functions. Observe that in general

$$\varepsilon_f(y,\pm,\lambda) + \varepsilon_b(y,\pm,\lambda) = 1 \tag{10.22}$$

for all depths y in the water body $X(x,z)$ and all $\lambda \in \Lambda$. The eccentricities ε_f and ε_b are such that, in natural hydrosols where there is marked forward scattering in the volume scattering function's σ-lobe, $\varepsilon_f(y,\pm,\lambda)$ tends to be somewhat larger than $\varepsilon_b(y,\pm,\lambda)$. In general we would expect in real media that

$$\varepsilon_b(y,\pm,\lambda) \leq \tfrac{1}{2} \leq \varepsilon_f(y,\pm,\lambda) \ . \tag{10.23}$$

The closer $\sigma(y;\underline{\xi}';\underline{\xi};\lambda)$ is to spherical shape when plotted as a function of $\underline{\xi}$ for fixed $\underline{\xi}'$, the closer are ε_f and ε_b to $\frac{1}{2}$, the midpoint of the unit interval [0,1]. If $p(y;\underline{\xi}';\underline{\xi};\lambda)$ is independent of $\underline{\xi}$ and equal to $(4\pi)^{-1}$, then we find from (10.20), (10,21) that $\varepsilon_f(y,\pm,\lambda) = \varepsilon_b(y,\pm,\lambda) = \frac{1}{2}$. It is expected that in real media $\varepsilon_f(y,\pm,\lambda)$ and $\varepsilon_b(y,\pm,\lambda)$ vary relatively little with y and $\pm$, and so may be given the status of apparent optical properties (H.O., vol. V, p. 106).

Next, the heterochromatic $\hat{f}$ and $\hat{b}$ functions factor into the forms

$$\hat{f}(y,\pm,\lambda',\lambda) = D(y,\pm,\lambda')\ \hat{s}(y,\lambda',\lambda)\ \hat{\varepsilon}_f(y,\pm,\lambda',\lambda) \tag{10.24}$$

$$\hat{b}(y,\pm,\lambda',\lambda) = D(y,\pm,\lambda')\ \hat{s}(y,\lambda',\lambda)\ \hat{\varepsilon}_b(y,\pm,\lambda',\lambda) \tag{10.25}$$

where $D(y,\pm,\lambda')$ are the distribution functions evaluated at the exciting or incident wavelength λ', and $\hat{s}(y,\lambda',\lambda)$ is the volume total transpectral scattering function defined in (4.4). The dimensionless numbers $\hat{\varepsilon}_f$ and $\hat{\varepsilon}_b$ are defined analogously to ε_f and ε_b:

$$'\hat{\varepsilon}_f(y,\pm,\lambda',\lambda)' \quad \text{for} \quad \frac{\int_{\Xi_\pm} \left[\int_{\Xi_\pm} N(y,\xi',\lambda')\ \hat{p}(y;\xi';\xi;\lambda',\lambda)\ d\Omega(\xi')\right] d\Omega(\underline{\xi})}{\int_{\Xi_\pm} N(y,\underline{\xi}',\lambda')\ d\Omega(\underline{\xi}')} \tag{10.26}$$

$$'\hat{\varepsilon}_b(y,\pm,\lambda',\lambda)' \quad \text{for} \quad \frac{\int_{\Xi_\mp} \left[\int_{\Xi_\pm} N(y,\xi',\lambda')\ \hat{p}(y;\xi';\xi;\lambda',\lambda)\ d\Omega(\underline{\xi}')\right] d\Omega(\underline{\xi})}{\int_{\Xi_\pm} N(y,\underline{\xi}',\lambda')\ d\Omega(\underline{\xi}')} \tag{10.27}$$

where $\hat{p}(y,\underline{\xi}';\underline{\xi};\lambda',\lambda)$ is the heterochromatic phase function defined in (7.3). We have in general

$$\hat{\varepsilon}_f(y,\pm,\lambda',\lambda) + \hat{\varepsilon}_b(y,\pm,\lambda',\lambda) = 1 \tag{10.28}$$

for all y, λ', and λ.

In real media $\hat{p}$, as a function of $\underline{\xi}$, is relatively spherical. In such a case $\hat{\varepsilon}_f$ and $\hat{\varepsilon}_b$ reduce to

$$\hat{\varepsilon}_f(y,\pm,\lambda',\lambda) = \hat{\varepsilon}_b(y,\pm,\lambda',\lambda) = \tfrac{1}{2} \tag{10.29}$$

and the expression for $\hat{f}$ and $\hat{b}$ in (10.24) and (10.25) simplify accordingly. In the interests of generality of the irradiance model in section 11, we will retain the full representations (10.24), (10.25) of $\hat{f}$ and $\hat{b}$ until specifically noted otherwise.

PART II. IRRADIANCE MODEL

11. IRRADIANCE TRANSFER MODEL, LOCAL INTERACTION PRINCIPLES

We now reduce the exact irradiance transfer equations (10.15) to a discrete approximate form ready for numerical work. The key step is to partition Λ into a finite set of intervals over which $H(y,\pm,\lambda)$, as functions of λ, are assumed to be constant. To be specific, we will limit the nonzero values of $H(y,\pm,\lambda)$ on $\Lambda = [0,\infty)$ to a finite subinterval $[\lambda_a,\lambda_b) \equiv \{\lambda: \ \lambda_a \leq \lambda < \lambda_b\}$ of Λ. For example we may take $\lambda_a = 400$ nm and $\lambda_b = 700$ nm. In general suppose we divide $\lambda_b-\lambda_a$ into m parts of arbitrary nonzero *band widths* $\Delta_1,\ldots,\Delta_m$, such that $\Delta_1 + \cdots + \Delta_m = \lambda_b-\lambda_a$. Set $\lambda_1 = \lambda_a$, $\lambda_2 = \lambda_1+\Delta_1$, and in general set $\lambda_{j+1} = \lambda_j + \Delta_j$, $j = 1,\ldots,m$, so that $\lambda_{m+1} = \lambda_b$. Next, represent $H(y,\pm,\lambda)$ for any λ in $[\lambda_a,\lambda_b)$ by the formal linear combination

$$H(y,\pm,\lambda) = \sum_{j=1}^{m} H(y,\pm,j)\ \chi(\lambda,j) \tag{11.1}$$

where, for $j = 1,\ldots,m$,

$$\chi(\lambda,j) = \begin{cases} 1 \text{ if } \lambda_j \leq \lambda \quad \text{and} \quad \lambda < \lambda_{j+1} \\ 0 \text{ if } \lambda < \lambda_j \quad \text{or} \quad \lambda_{j+1} \leq \lambda \end{cases} \tag{11.2}$$

i.e., $\chi(\lambda,j)$ is 1 if $\lambda \in \Lambda_j$ $(\equiv [\lambda_j,\lambda_{j+1}))$ and 0 if $\lambda \notin \Lambda_j$. The number $H(y,\pm,j)$, (with units $W\cdot m^{-2}\cdot nm^{-1}$) for each y, and stream flow ($\pm$) is the average of $H(y,\pm,\lambda)$ over Λ_j and is considered unknown. The $H(y,\pm,j)$, $j = 1,\ldots,m$ are to be determined by solving the discrete forms of (10.15) now under derivation. The discrete forms of (10.15) are obtained by averaging each side of (10.15) over each Λ_j, $j = 1,\ldots,m$. For example, the average of the left side of (10.15) over Λ_j becomes, on using the representation (11.1),

$$\mp \Delta_j^{-1} \int_{\Lambda_j} \frac{dH(y,\pm,\lambda)}{dy} \, d\lambda = \mp \frac{dH(y,\pm,j)}{dy} \quad , \quad j = 1,\ldots,m \tag{11.3}$$

This averaging operation applied to the first term on the right of (10.15) yields

$$- \Delta_j^{-1} \int_{\Lambda_j} [a(y,\pm,\lambda) + b(y,\pm,\lambda)] \sum_{j=1}^{m} H(y,\pm,j) \, \chi(\lambda,j) \, d\lambda$$

$$= \{- \Delta_j^{-1} \int_{\Lambda_j} [a(y,\pm,\lambda) + b(y,\pm,\lambda)] \, d\lambda\} \, H(y,\pm,j) \quad , \quad j = 1,\ldots,m \tag{11.4}$$

Now, by the mean value theorem for integration, since $D(y,\pm,\lambda)$ and $a(y,\lambda)$ are non-negative valued functions over the ranges of all their arguments, and over the λ-range in particular, we find,

$$\begin{aligned} \Delta_j^{-1} \int_{\Lambda_j} a(y,\pm,\lambda) \, d\lambda &= \Delta_j^{-1} \int_{\Lambda_j} D(y,\pm,\lambda) \, a(y,\lambda) \, d\lambda \\ &= D(y,\pm,j) \, \Delta_j^{-1} \int_{\Lambda_j} a(y,\lambda) \, d\lambda \\ &\equiv D(y,\pm,j) \, \bar{a}(y,j) \end{aligned} \tag{11.5}$$

where we have defined $\bar{a}(y,j)$ in context.

Thus there is, for fixed y, some value of $D(y,\pm,\lambda)$, say $D(y,\pm,j)$, on the sub interval Λ_j of the spectrum Λ that can be drawn out of the integrand, leaving $a(y,\lambda)$ to be averaged over Λ_j. In real media we expect the λ behavior of $D(y,\pm,\lambda)$ to be relatively mild over Λ_j, compared to that of $a(y,\lambda)$; hence our choice of drawing $D(y,\pm,j)$ out of the integral.

In a similar way we reduce the b-term in (11.4). On recalling (10.19), we can write

$$'\overline{b}(y,j)' \text{ for } \varepsilon_b(y,\pm,j)\ \Delta_j^{-1} \int_{\Lambda_j} s(y,\lambda)\, d\lambda \qquad (m^{-1}) \tag{11.6a}$$

Then

$$\Delta_j^{-1} \int_{\Lambda_j} b(y,\pm,\lambda)\, d\lambda \equiv D(y,\pm,j)\ \overline{b}(y,j) \tag{11.6b}$$

It is easy to show that $D(y,\pm,j)$ found in this way can be common to both the a and b terms in (11.4). Let us write, for $j = 1,\ldots,m$,

$$'\tau(y,\pm|j,j)' \quad \text{for} \quad -D(y,\pm,j)\ [\overline{a}(y,j) + \overline{b}(y,j)] \qquad (m^{-1}) \tag{11.7}$$

This is the *local transmittance function* for the monochromatic case, $x \le y \le z$.

We may next treat the monochromatic backscatter term in (10.15) in a similar manner. Thus for $j = 1,\ldots,m$, we have, as in (11.6)

$$\begin{aligned} \Delta_j^{-1} \int_{\Lambda_j} b(y,\mp,\lambda)\, d\lambda &\equiv D(y,\mp,j)\ \overline{b}(y,j) \\ &\equiv \rho(y,\pm|j,j) \quad , \quad j = 1,\ldots,m. \end{aligned} \tag{11.8}$$

where D and $\overline{b}$ are as defined above, and where we write

$$'\rho(y,\pm|j,j)' \quad \text{for} \quad D(y,\pm,j)\ \overline{b}(y,j) \tag{11.9}$$

as the *local reflectance function* for the monochromatic case, $x \le y \le z$. We give $\tau(y,\pm|j,j)$ in (11.7) and $\rho(y,\pm|j,j)$ in (11.8) the transmittance and reflectance connotations for reasons that will become clear when the irradiance model takes its final form below.

We come next to the integrals on the right in (10.15). We reduce the forward-scatter integral averaged over Λ_j, using (11.1) and (10.24) as follows:

$$\Delta_j^{-1} \int_{\Lambda_j} [\int_{\Lambda} \hat{f}(y,\pm,\lambda',\lambda) \sum_{i=1}^{m} H(y,\pm,i)\ \chi(\lambda',i)\ d\lambda']\ d\lambda \quad , \quad i,j = 1,\ldots m,\ i \neq j$$

$$= \sum_{i=1}^{m} \Delta_j^{-1} \int_{\Lambda_j} [\int_{\Lambda_i} \hat{f}(y,\pm,\lambda',\lambda)\ d\lambda']\ d\lambda\ H(y,\pm,i)$$

$$\cong \sum_{i=1}^{m} D(y,\pm,i)\ \varepsilon_f(y,\pm,i)\ \Delta_j^{-1} \int_{\Lambda_j} [\int_{\Lambda_j} \hat{s}(y,\lambda',\lambda)\ d\lambda']\ d\lambda\ H(y,\pm,i)$$

$$\equiv \sum_{i=1}^{m} D(y,\pm,i)\ \varepsilon_f(y,\pm,i)\ \bar{s}(y,i,j)\ H(y,\pm,i)\ , \qquad (11.10)$$

where we have written, for $i \neq j$:

$$'\bar{s}(y,i,j)' \quad \text{for} \quad \Delta_j^{-1} \int_{\Lambda_j} [\int_{\Lambda_j} \hat{s}(y,\lambda',\lambda)\ d\lambda']\ d\lambda \qquad (m^{-1}) \qquad (11.11)$$

By our convention on $\hat{\sigma}$ in (4.3) we set

$$\bar{s}(y,i,i) \equiv 0 \quad , \quad \text{for all } y \text{ and } i = 1,\ldots,m. \qquad (11.11a)$$

Moreover, we have used the mean value of $D(y,\pm,i)$ defined above (hence the sign '$\cong$' in the next to last line in (11.10)) and then have used the mean value theorem for integration to draw out the mean value $\varepsilon_f(y,\pm,i)$ from the λ' integration, leaving $\hat{s}(y,\lambda',\lambda)$ to be averaged as shown. $\bar{s}(y,i,j)$ is defined as the result of the averaging, as shown in (11.11).

In an exactly similar way we treat the average over Λ_j of the backscatter integral term in (10.15) to find the following backscatter counterpart to (11.10):

$$\Delta_j^{-1} \int_{\Lambda_j} [\int_{\Lambda} \hat{b}(y,\pm,\lambda',\lambda)\ H(y,\pm,\lambda')\ d\lambda']\ d\lambda \quad , \quad i,j = 1,\ldots,m, \quad i \neq j.$$

$$= \sum_{i=1}^{m} D(y,\pm,i)\ \varepsilon_b(y,\pm,i)\ \bar{s}(y,i,j)\ H(y,\pm,i)\ . \qquad (11.12)$$

Further, let us write, for $i,j = 1,\ldots,m,\ i \neq j$,

$$\text{'}\tau(y,\pm|i,j)\text{'} \quad \text{for} \quad D(y,\pm,i)\ \varepsilon_f(y,\pm,i)\ \bar{s}(y,i,j) \qquad (m^{-1})$$

$$(11.13)$$

$$\text{'}\rho(y,\pm|i,j)\text{'} \quad \text{for} \quad D(y,\pm,i)\ \varepsilon_b(y,\pm,i)\ \bar{s}(y,i,j) \qquad (m^{-1})$$

These are the *local transpectral transmittance and reflectance functions* $x \le y \le z$. It is very likely that in fluorescent light, $D(y,\pm,i) \cong 2$, and hence that $\varepsilon_f \cong \varepsilon_b \cong ½$; however, for now we retain ε_f and ε_b in their general forms.

Finally, the true emission (source density) term in (10.15) becomes, by definition

$$h_e(y,\pm,j) \equiv \Delta_j^{-1} \int_{\Lambda_j} h_e(y,\pm,\lambda)\ d\lambda \quad , \ j = 1,\ldots,m \qquad (W\cdot m^{-3}\cdot nm^{-1}) \quad (11.14)$$

Assembling and summarizing all these results, we obtain the desired discrete form of the irradiance model (10.15) for fluorescent light fields:

$$\mp \frac{dH(y,\pm,j)}{dy} = \sum_{i=1}^{m} H(y,\pm,i)\ \tau(y,\pm|i,j) + \sum_{i=1}^{m} H(y,\mp,i)\ \rho(y,\mp|i,j) + h_e(y,\pm,j)$$

$$x \le y \le z,\ j = 1,\ldots,m. \qquad (11.15)$$

where y is depth (in m).

This may be placed into matrix form. First define the irradiance vectors

$$'\underline{H}(y,\pm)' \quad \text{for} \quad [H(y,\pm,1),\ldots,H(y,\pm,m)] \qquad (W\cdot m^{-2}\cdot nm^{-1}) \qquad (11.16)$$

$$'\underline{h}_e(y,\pm)' \quad \text{for} \quad [h_e(y,\pm,1),\ldots,h_e(y,\pm,m)] \quad (W\cdot m^{-3}\cdot nm^{-1})$$

and then the *local transmittance* and *local reflectance matrices* over the depth range $x \le y \le z$:

$$'\underline{\tau}(y,\pm)' \quad \text{for} \quad \begin{bmatrix} \tau(y,\pm|1,1) & \tau(y,\pm|1,2) & \cdots & \tau(y,\pm|1,m) \\ \tau(y,\pm|2,1) & \tau(y,\pm|2,2) & \cdots & \tau(y,\pm|2,m) \\ \vdots & \vdots & & \vdots \\ \tau(y,\pm|m,1) & \tau(y,\pm|m,2) & \cdots & \tau(y,\pm|m,m) \end{bmatrix} \quad (m^{-1}) \quad (11.17)$$

$$'\underline{\rho}(y,\pm)' \quad \text{for} \quad \begin{bmatrix} \rho(y,\pm|1,1) & \rho(y,\pm|1,2) & \cdots & \rho(y,\pm|1,m) \\ \rho(y,\pm|2,1) & \rho(y,\pm|2,2) & \cdots & \rho(y,\pm|2,m) \\ \vdots & \vdots & & \vdots \\ \rho(y,\pm|m,1) & \rho(y,\pm|m,2) & \cdots & \rho(y,\pm|m,m) \end{bmatrix} \quad (m^{-1}) \quad (11.18)$$

The matrix form of (11.15) is then (with depth y in meters)

$$\mp \frac{d\underline{H}(y,\pm)}{dy} = \underline{H}(y,\pm)\ \underline{\tau}(y,\pm) + \underline{H}(y,\mp)\ \underline{\rho}(y,\mp) + \underline{h}_e(y,\pm) \qquad (11.19)$$

$$x \le y \le z$$

or more compactly

$$\boxed{\frac{d\underline{H}(y)}{dy} = \underline{H}(y)\,\underline{\mathcal{K}}(y) + \underline{h}_e(y)} \tag{11.19a}$$

where we have written

$$'\underline{H}(y)' \quad \text{for} \quad [\underline{H}(y,+),\ \underline{H}(y,-)]\ ,$$

$$'\underline{h}_e(y)' \quad \text{for} \quad [-\underline{h}_e(y,+),\ \underline{h}_e(y,-)]$$

$$'\underline{\mathcal{K}}(y)' \quad \text{for} \quad \begin{bmatrix} -\underline{\tau}(y,+) & \underline{\rho}(y,+) \\ -\underline{\rho}(y,-) & \underline{\tau}(y,-) \end{bmatrix}$$

Equations (11.19) express the *local interaction principles*. From these spring the global interaction principles to be developed in section 14. When fluorescence is absent, then (11.19) reduces to a set of independent equations, one for each wavelength interval. The associated boundary conditions to (11.19) follow at once from the λ-averaged versions of the boundary conditions (10.16). Under our present assumptions we have a nonfluorescing infinitesimally thin air-water boundary X[a,x]. See the derivation and discussion of (8.2). Moreover, the lower boundary X(z,b) will be considered nonfluorescing and either of a matte-surface form or a water slab of finite thickness with a matte reflecting bottom. The matrix versions of (10.16) are then

$$\text{for } X(a,x): \quad \boxed{\begin{aligned} \underline{H}(a,+) &= \underline{H}(x,+)\,\underline{t}(x,a) + \underline{H}(a,-)\,\underline{r}(a,x) \qquad (11.20a)\\ \underline{H}(x,-) &= \underline{H}(x,+)\,\underline{r}(x,a) + \underline{H}(a,-)\,\underline{t}(a,x) \qquad (11.20b)\end{aligned}}$$

and

$$\text{for } X(z,b): \quad \boxed{\begin{aligned} \underline{H}(z,+) &= \underline{H}(b,+)\,\underline{t}(b,z) + \underline{H}(z,-)\,\underline{r}(z,b) \qquad (11.20c)\\ \underline{H}(b,-) &= \underline{H}(b,+)\,\underline{r}(b,z) + \underline{H}(z,-)\,\underline{t}(z,b) \qquad (11.20d)\end{aligned}}$$

where we write

$$\begin{aligned} &(\underline{t}_+ =)\ '\underline{t}(x,a)' \quad \text{for} \quad \text{diag}[t(x,a,1),\ldots,t(x,a,m)]\\ &(\underline{r}_- =)\ '\underline{r}(a,x)' \quad \text{for} \quad \text{diag}[r(a,x,1),\ldots,r(a,x,m)] \end{aligned} \qquad (11.21)$$

The remaining pair $\underline{r}_+,\underline{t}_-$ of diagonal matrices for X(a,x) are found following the (a,x)-interchange change rule (see discussion of (10.17). The quartet of diagonal matrices $\underline{r}_\pm,\underline{t}_\pm$ for X(z,b) are structured similarly.

The basic problem of the fluorescent irradiance model may now be stated as follows:

Given: A plane-parallel optical medium X(a,b) as in Fig. 7, with heterochromatic optical properties $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$ and $\underline{h}_e(y,\pm)$ as in (11.19), continuously varying over the water body X(x,z) of X(a,b), $x \le y \le z$, with given $\underline{r}_\pm,\underline{t}_\pm$ for the upper boundary X(a,x) as in (10.20), and lower boundary X(z,b) of X(a,b), and also given incident irradiances $\underline{H}(a,-)$, $\underline{H}(b,+)$.

Required: The heterochromatic internal irradiance vectors $\underline{H}(y,\pm)$, $x \le y \le z$ throughout X(x,z) and the emergent boundary irradiance vectors $\underline{H}(a,+)$, $\underline{H}(b,-)$.

The solutions of (11.19) subject to (11.20) provide the direct solution of this problem. These solutions will be considered in Part III. See in particular sections 19, 20 and 21. The inverse solution is considered in Part IV, below.

For the remainder of Part II we shall look at the subject of directly observable properties of the fluorescent light field. These are traditionally useful properties that, by definition, can be obtained without numerically solving the system (11.19), (11.20).

12. DIRECTLY OBSERVABLE RELATIONS

The local interaction principles (11.19) contain useful information about fluorescent light fields in natural hydrosols that may be elicited without having to first numerically solve that system of equations. We shall now exhibit some of this information and point out where it may be of use. In doing so we will be extending to the fluorescent level various results in the theory of directly observable relations originally developed in the monochromatic setting for arbitrarily stratified media (cf. Preisendorfer, 1961), and which now may be found in H.O., vol. V, sec. 9.2.

A. *Observable Reflectance Matrix*

Of some interest in the study of fluorescent light fields in stratified lakes and seas is the search, by aerial surveys, for the presence and extent of chlorophyll activity. The presence of fluorescent chlorophyll activity in source-free media, i.e., those in which the emission term $h_e(y,\pm,j)$ of (11.15) is zero, can be monitored by its fluorescing in the red end of the visible spectrum. Natural waters free of such fluorescence activity would exhibit upward irradiances $H(y,+,\lambda)$ at depths y below the surface that are related to the downward radiance $H(y,-,\lambda)$ at level y of the same wavelength λ by the simple relation

$$H(y,+,\lambda) = H(y,-,\lambda)\ R(y,-,\lambda) \tag{12.1}$$

where $R(y,-,\lambda)$ is the (dimensionless) *irradiance ratio* of the monochromatic theory (H.O., Vol. V, p. 116). In source-free media, but in the presence of fluorescing materials, downward irradiance $H(y,-,\lambda')$ for wavelengths λ' mainly shorter than λ will by transpectral scattering enhance the upward irradiance

$H(y,+,\lambda)$ beyond the value given by (12.1). To handle this possibility one introduces the *transpectral irradiance ratio* $R(y,-,\lambda',\lambda)$ (in units of nm^{-1}) at depth y, and defines it via the relation

$$H(y,+,\lambda) = \int_{\Lambda} H(y,-,\lambda')\ R(y,-,\lambda',\lambda)\ d\lambda' \qquad (12.2)$$

for all y in $[x,z]$ and λ in $\Lambda = [0,\infty)$ or $[\lambda_a,\lambda_b]$, as the case may be. The ontological basis for $R(y,-,\lambda',\lambda)$ in source-free stratified media rests in the interaction principle of Hydrologic Optics (H.O., vol. II, p. 205 and p. 279). In the context of the source-free irradiance model (11.19) this relation takes the discretized form

$$H(y,+,j) = \sum_{i=1}^{m} H(y,-,i)\ R(y,-,i,j) \qquad (12.3)$$

for $x \le y \le z$ and $j = 1,\ldots,m$, where we have used (11.1) in the averaged equation (12.2) and have written

$$\text{'}R(y,-|i,j)\text{'} \quad \text{for} \quad \Delta_j^{-1} \int_{\Lambda_j} \left[\int_{\Lambda_i} R(y,-,\lambda',\lambda)\ d\lambda'\right] d\lambda \qquad (12.4)$$

for $i,j = 1,\ldots,m$. Observe that $R(y,-,i,j)$ is dimensionless. When there is no transpectral scatter, then $R(y,-,i,j) = 0$ for $i \neq j$, as we shall see rigorously below in Eq. (12.11). In this case $R(y,-,i,i)$ becomes the $R(y,-,\lambda)$ of the monochromatic theory in H.O., Vol. V., for some specified wavelength λ in Λ_j, $j = 1,\ldots,m$. The problem of determining the enhanced reflectance in the red part of the spectrum, and hence the problem of detecting chlorophyll activity, can be solved exactly if we can model the flow of radiant energy across the spectrum via the irradiances $H(y,\pm,j)$, $j = 1,\ldots,m$.

Towards this end, the relation (12.3) in source-free media can be placed into the compact matrix form

$$\underline{H}(y,+) = \underline{H}(y,-)\ \underline{R}(y,-) \tag{12.5}$$

where $\underline{H}(y,\pm)$ are defined in (11.16) and where we have written

$$'\underline{R}(y,-)' \quad \text{for} \quad \begin{bmatrix} R(y,-|1,1) & R(y,-|1,2) & \cdots & R(y,-|1,m) \\ R(y,-|2,1) & R(y,-|2,2) & \cdots & R(y,-|2,m) \\ \vdots & \vdots & & \vdots \\ R(y,-|m,1) & R(y,-|m,2) & \cdots & R(y,-|m,m) \end{bmatrix} \tag{12.6}$$

for all y such that $x \le y \le z$. Henceforth in this section we will assume X(a,b) is finitely deep and source-free,* i.e., $h_e = 0$ in (11.15).

Whenever the $m \times m$ matrix $\underline{R}(y,-)$ is assumed or demonstrated to be invertible, we will write

$$'\underline{R}(y,+)' \quad \text{for} \quad \underline{R}^{-1}(y,-) \tag{12.7}$$

and then (12.5) has the reciprocal companion

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* This does not constitute any loss of generality in the present theory. It will be shown in section 15 how the presence of true sources in the heterochromatic model can be handled whenever they are present in X(a,b). Observe also that the source-free equations (11.15) can be rearranged so as to take on the appearance of the two-flow irradiance equations for monochromatic flux. The resultant equations for $H(y,\pm,j)$ will have a source term, namely the term into which all the remaining transpectral terms (for $i \ne j$) have been swept. Hence in the presence of fluorescence the source-free heterochromatic irradiance model corresponds to a monochromatic model with source term.

$$\underline{H}(y,-) = \underline{H}(y,+)\ \underline{R}(y,+) \tag{12.8}$$

We now make some preliminary observations about $\underline{R}(y,\pm)$, and some of its special cases.

If we set y equal to z, in $\underline{R}(y,-)$, where z is the depth of the lower nonfluorescing boundary, and set $\underline{H}(b,+) = \underline{0}$, then $\underline{R}(z,-)$ is diagonal and we can set it equal to $\underline{r}(z,b)$ in (11.20c). (Recall the discussion of the boundary reflectances in (8.2c,d) in which the special nonfluorescing form of X(z,b) is adopted.) Similarly, $\underline{R}(x,+)$ reduces to $\underline{r}(x,a)$ when we set $\underline{H}(a,-) = 0$ in (11.20b). In natural media the condition $\underline{H}(b,+) = \underline{0}$ is normally satisfied, while $\underline{H}(a,-) = \underline{0}$ is not, since light enters from above but not from below. When $\underline{H}(b,+) = \underline{0}$, then $\underline{R}(y,-)$, and more generally $R(y,-,\lambda',\lambda)$ in (12.2), can be made to rest on the interaction principle and behaves like a global optical property (this will be shown in (12.12), below). Hence in what follows we will work more with $\underline{R}(y,-)$ (rather than $\underline{R}(y,+)$) and draw out the directly observable relations based on $\underline{R}(y,-)$. It should be noted, however, that $\underline{R}(y,+)$ in general and by definition formally exists whenever $\underline{R}^{-1}(y,-)$ does, for any y in [x,z], regardless of boundary conditions. It is simply that whenever $\underline{H}(a,-) \neq \underline{0}$, $\underline{R}(y,+)$ does not (nor does its monochromatic counterpart R(y,+) in H.O., vol. V, sec. 9.2) take on the meaning of an *irradiance reflectance* (i.e., in the sense of an interaction-principle operator, i.e., an operator that occurs in the global interaction principles; cf. sec. 15) at depth y below the air-water surface X(a,x). Thus, whenever $\underline{H}(a,-) \neq 0$ or $\underline{H}(b,+) \neq 0$ hold, and we wish to determine the light field in X(a,b), we will set aside $\underline{R}(y,+)$ or $\underline{R}(y,-)$, respectively, and use the global reflectance operators $\underline{R}(y,x)$ or $\underline{R}(y,z)$, respectively, as defined in section 15. With these observations in mind, we now proceed to extend the classical theory of monochromatic observable optical properties of R(y,-), and other two-flow irradiance concepts to the heterochromatic case.

B. *Differential Equation for* $\underline{R}(y,-)$

The basic relation (12.5) connecting the upwelling irradiance $\underline{H}(y,+)$ with the downwelling irradiance $\underline{H}(y,-)$ at depth y (in meters) allows in particular the computation of $\underline{H}(y,+)$ when $\underline{H}(y,-)$ is known along with $\underline{R}(y,-)$. In practice we would know or estimate $\underline{H}(y,-)$. We therefore require a way to determine $\underline{R}(y,-)$. It will now be shown that $\underline{R}(y,-)$ obeys a matrix Riccati differential equation that, on using the initial condition $\underline{R}(z,-) = \underline{r}(z,b)$, may be integrated from level z to any level y, $x \le y \le z$, knowing the depth dependences of the local matrices $\underline{\rho}(y,\pm)$ and $\underline{\tau}(y,\pm)$ along the way.

Thus, on differentiating each side of (12.5) we find

$$\frac{d\underline{H}(y,+)}{dy} = \frac{d\underline{H}(y,-)}{dy}\, \underline{R}(y,-) + \underline{H}(y,-)\, \frac{d\underline{R}(y,-)}{dy} \tag{12.9}$$

Replacing the $\underline{H}$-derivatives by their algebraic expressions given in the source-free form of (11.19), we find

$$\{-\underline{H}(y,+)\,\underline{\tau}(y,+) - \underline{H}(y,-)\,\underline{\rho}(y,-)\} = \{\underline{H}(y,-)\,\underline{\tau}(y,-) + \underline{H}(y,+)\,\underline{\rho}(y,+)\}\,\underline{R}(y,-) + \underline{H}(y,-)\,\frac{d\underline{R}(y,-)}{dy} \tag{12.10}$$

Using (12.5) to replace $\underline{H}(y,+)$ by $\underline{H}(y,-)\underline{R}(y,-)$, and rearranging terms, we have, since $\underline{H}(y,-)$ is generally arbitrary,

$$\boxed{\begin{aligned} -\frac{d\underline{R}(y,-)}{dy} &= [\underline{\rho}(y,-) + \underline{R}(y,-)\,\underline{\tau}(y,+)] + [\underline{\tau}(y,-) + \underline{R}(y,-)\,\underline{\rho}(y,+)]\,\underline{R}(y,-) \\ &= \underline{R}(y,-)\,[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{R}(y,-)] + [\underline{\rho}(y,-) + \underline{\tau}(y,-)\,\underline{R}(y,-)] \\ x \le y \le z & \qquad\qquad (12.11) \end{aligned}}$$

with initial condition

$$\underline{R}(z,-) = \underline{r}(z,b) \qquad (12.11a)$$

This is the required Riccati differential equation for $\underline{R}(y,-)$. When fluorescence is absent, the $m \times m$ local reflectance and transmittance matrices $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$ of (11.17), (11.18) become diagonal, and then so also does $\underline{R}(y,-)$; hence (12.5) reduces to separate scalar equations of the form (12.1). See H.O., vol. V, Eq. (1), p. 148 for the monochromatic case to which (12.11) returns when the medium is fluorescence-free.

Integrating (12.11) from depth z to depth $y < z$, starting with (12.11a) yields the desired matrix $\underline{R}(y,-)$, $x \le y \le z$, knowing the $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$ along the way.

For completeness, we write down the differential equation for $\underline{R}(y,+)$, the companion to (12.11). That this, and even (12.11), is at all possible (in view of our comments about $\underline{R}(y,\pm)$ above) stems from the connections between either $\underline{R}(y,+)$ or $\underline{R}(y,-)$ and its more fundamental counterpart $\underline{R}(y,x)$ or $\underline{R}(y,z)$, respectively. These latter reflectances, which are associated with well-defined slabs $X[x,y]$, $X[y,z]$, respectively, will be introduced and studied in sections 15 and 16. The derivation of (12.12) below proceeds with all details the same as that of (12.11); the result is

$$\frac{d\underline{R}(y,+)}{dy} = [\underline{\rho}(y,+) + \underline{R}(y,+)\ \underline{\tau}(y,-)] + [\underline{\tau}(y,+) + \underline{R}(y,+)\ \underline{\rho}(y,-)]\ \underline{R}(y,+)$$

$$= \underline{R}(y,+)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,+)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\ \underline{R}(y,+)]$$

$$x \le y \le z \qquad (12.12)$$

with initial condition

$$\underline{R}(x,+) = r(x,a) \qquad (12.12a)$$

C. *The $\underline{K}$-functions for $\underline{H}(y,\pm)$*

We next return to the monochromatic theory and generalize to the heterochromatic case the logarithmic derivatives of $H(y,\pm)$:

$$K(y,\pm) \equiv \frac{-1}{H(y,\pm)} \cdot \frac{dH(y,\pm)}{dy} \qquad (12.13)$$

These K-functions allow us to visualize the depth dependence of $H(y,\pm)$ as a generalized form of exponential decay:

$$H(y,\pm) = H(x,\pm)\ \exp\{-\int_x^y K(u,\pm)du\} \qquad (12.14)$$

In the monochromatic case we have for $K(y,\pm)$ the expressions (cf. H.O., vol. V, p. 117):

$$K(y,-) = [a(y,-) + b(y,-)] - b(y,+)\ R(y,-) \qquad (m^{-1}) \qquad (12.15)$$

$$-K(y,+) = [a(y,+) + b(y,+)] - b(y,-)\ R(y,+) \qquad (m^{-1}) \qquad (12.16)$$

over the y-depth range $[x,y]$. In infinitely deep homogeneous media $a(y,\pm)$, $b(y,\pm)$ and hence $R(y,-)$ are independent of y and so then is $K(y,-) = k_-$ (cf. H.O., vol. V, p. 81). Eq. (12.14) then becomes

$$H(y,\pm) = H(x,\pm)\exp\{-k_-(y-x)\} \tag{12.17}$$

To find the K-function, say $\underline{K}(y,-)$, in the fluorescent setting, we write out the downward member of the source-free version of (11.19):

$$\frac{d\underline{H}(y,-)}{dy} = H(y,-)\,\underline{\tau}(y,-) + \underline{H}(y,+)\,\underline{\rho}(y,+)$$

and use the definition (12.5) for $\underline{R}(y,-)$. After rearrangement we have

$$\boxed{\frac{d\underline{H}(y,-)}{dy} = \underline{H}(y,-)\,[\underline{\tau}(y,-) + \underline{R}(y,-)\,\underline{\rho}(y,+)]} \tag{12.18}$$

On this basis we write

$$\text{'}\underline{K}(y,-)\text{'} \text{ for } -[\underline{\tau}(y,-) + \underline{R}(y,-)\,\underline{\rho}(y,+)] \qquad (m^{-1}) \tag{12.19}$$

From the upward member of the source-free version of (11.19) along with (12.8) we are led to write

$$\text{'}-\underline{K}(y,+)\text{'} \text{ for } -[\underline{\tau}(y,+) + \underline{R}(y,+)\,\underline{\rho}(y,-)] \qquad (m^{-1}) \tag{12.20}$$

which yields the differential equation for $\underline{H}(y,+)$:

$$\boxed{-\frac{d\underline{H}(y,+)}{dy} = \underline{H}(y,+)\,[\underline{\tau}(y,+) + \underline{R}(y,+)\,\underline{\rho}(y,-)]} \tag{12.21}$$

In practice, if we have determined $\underline{R}(y,-)$ over the y-range [x,z], then from (12.18), with initial value H(x,-), we can integrate (12.18) for $\underline{H}(y,-)$ to any depth y in the slab X(x,z). Thus, by (12.14) and (12.19) we can formally evaluate H(y,-) in

$$\underline{H}(y,-) = \underline{H}(x,-) \exp\{ \int_x^y [\underline{\tau}(u,-) + \underline{R}(u,-)\ \underline{\rho}(u,+)]\ du\} \tag{12.22}$$

In practice we can use (12.18) directly to find $\underline{H}(y,-)$, with some numerical matrix differential equation subroutine. One way to accomplish this would be to first integrate (12.11) from z to x starting with $\underline{r}(z,b)$ and ending with $\underline{R}(x,-)$. Then integrate (12.18) downward from x to a general level y starting with H(x,-), and the 'initial' value of $\underline{R}(y,-)$, namely $\underline{R}(x,-)$. As the integration of (12.18) proceeds, 'deintegrate' (12.11) (in the obvious sense) simultaneously, so that $\underline{R}(y,-)$ is available at each y for use in (12.11). This eliminates the need to store all the $\underline{R}(y,-)$ values obtained in the upward sweep. This idea will be systematically explored in section 19.* Finally, at each depth y, we find $\underline{H}(y,+)$ by evaluating $\underline{H}(y,-)\ \underline{R}(y,-)$.

D. *Alternative Differential Equation for* $\underline{R}(y,-)$

The monochromatic two-flow theory of the light field (H.O., vol. V, p. 123) shows that the depth rate of change of R(y,-) can be characterized by

$$\frac{dR(y,-)}{dy} = R(y,-)[K(y,-) - K(y,+)] \tag{12.23}$$

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* An obvious alternative to (12.18) is (11.19). After integrating (12.11) from z to x, we have R(x,-) and hence H(x,+) = H(x,-)R(x,-). The pair of values $H(x,\pm)$ then are the initial values to start integrating (11.19), with $\underline{h}_e(y,\pm) = \underline{0}$. This procedure, however, is numerically unstable. See Preisendorfer, 1976a.

This shows in particular that $R(y,-)$ is independent of y on some depth interval if and only if $K(y,-) = K(y,+)$ over that depth interval. The heterochromatic generalization of (12.23) is

$$\frac{d\underline{R}(y,-)}{dy} = \underline{K}(y,-)\ \underline{R}(y,-) - \underline{R}(y,-)\ \underline{K}(y,+) \tag{12.24}$$

which follows at once from (12.11) and the pair of definitions (12.19), (12.20).

E. *The* $\underline{R}$-$\underline{K}$ *Connections*

In the monochromatic theory (cf. H.O., vol. V, p. 117), we can represent $R(y,-)$ in terms of $\rho(y,\pm)$, $\tau(y,\pm)$, and $K(y,\pm)$ at each depth y by

$$\begin{aligned} R(y,-) &= \frac{\rho(y,-)}{[K(y,+) - \tau(y,+)]} \\ &= -\frac{[K(y,-) + \tau(y,-)]}{\rho(y,+)} \end{aligned} \tag{12.25}$$

The heterochromatic counterparts to these follow from (12.19), (12.20) and the definition (12.7):

$$\boxed{\begin{aligned} \underline{R}(y,-) &= \underline{\rho}(y,-)\ [\underline{K}(y,+) - \underline{\tau}(y,+)]^{-1} \\ &= -[\underline{K}(y,-) + \underline{\tau}(y,-)]\underline{\rho}^{-1}(y,+) \end{aligned}} \tag{12.26}$$

The usefulness of either of these sets of representations occurs in homogeneous infinitely deep layers wherein the quantities on the right sides are independent of depth y. Then $\underline{R}(y,-)$ is also independent of y and is characterized by the local properties ρ, τ, K (resp. $\underline{\rho}$, $\underline{\tau}$, $\underline{K}$) of the medium (cf. (15.69) and (15.70)).

F. *Divergence Relation*

In the monochromatic theory (cf. H.O., vol. I, p. 62) the divergence relation with source term is

$$-\frac{d\bar{H}(y,+)}{dy} = \nabla\cdot\underline{H}(y) = -a(y)h(y) + h_e(y) \tag{12.27}$$

In the heterochromatic theory, this is generalized to (9.5). The matrix version of (9.5) is obtained by observing that we can write $h(y,\lambda)$ as $D(y,+,\lambda)H(y,+,\lambda) + D(y,-,\lambda)H(y,-,\lambda)$. Use (11.1) and average each side of (9.5) over Λ_j. Use (11.5) and (11.11), and write '$h(y,\pm,j)$' for $D(y,\pm,i)H(y,\pm,i)$. Then we obtain

$$\boxed{-\frac{d\underline{\bar{H}}(y,+)}{dy} = \nabla\cdot\underline{H}(y) = \underline{h}(y)\,[\underline{\hat{s}}(y) - \underline{a}(y)] + \underline{h}_e(y)} \tag{12.28}$$

where we have written

$$\text{'}\underline{\hat{s}}(y)\text{'} \quad \text{for} \quad \begin{bmatrix} 0 & \bar{s}(y,1,2) & \cdots & \bar{s}(y,1,m) \\ \bar{s}(y,2,1) & 0 & \cdots & \bar{s}(y,2,m) \\ \vdots & \vdots & & \vdots \\ \bar{s}(y,m,1) & \bar{s}(y,m,2) & \cdots & 0 \end{bmatrix} \tag{12.29}$$

and $$\text{'}\underline{a}(y)\text{'} \quad \text{for} \quad \mathrm{diag}[\bar{a}(y,1),\ldots,\bar{a}(y,m)] \qquad (m^{-1}) \tag{12.30}$$

Further, write

$$'\underline{\bar{H}}(y,+)' \quad \text{for} \quad [\bar{H}(y,+,1),\ldots,\bar{H}(y,+,m)] \qquad (W\cdot m^{-2}\cdot nm^{-1})$$

$$'\underline{h}(y)' \quad \text{for} \quad [h(y,1),\ldots,h(y,m)] \qquad (W\cdot m^{-2}\cdot nm^{-1}) \tag{12.31}$$

$$'\underline{h}_e(y)' \quad \text{for} \quad [h_e(y,1),\ldots,h_e(y,m)] \qquad (W\cdot m^{-3}\cdot nm^{-1})$$

Here $\bar{H}(y,+,j)$, $h(y,j)$ and $h_e(y,j)$ are, respectively, the discrete forms of $\bar{H}(y,+,\lambda)$, $h(y,\lambda)$ and $h_e(y,\lambda)$ in (9.5). Observe that if there is no transpectral scattering, then $\underline{\hat{s}}(y) = \underline{0}$, and (12.28) splits into m copies of (12.27).

G. *The Basic Reflectance Relation*

In the monochromatic two-flow theory of the light field in stratified media, the reflectance function is representable (cf. H.O., vol. V, p. 118) in terms of a and K as

$$R(y,-) = \frac{K(y,-) - a(y,-)}{K(y,+) + a(y,+)} \tag{12.32}$$

The present heterochromatic version of this is

$$\boxed{\underline{R}(y,-) = [\underline{K}(y,-) - \underline{A}(y,-)][\underline{K}(y,+) + \underline{A}(y,+)]^{-1}} \tag{12.33}$$

where $\underline{K}(y,\pm)$ are as given in (12.19), (12.20), and where we have written

$$'\underline{A}(y,\pm)' \quad \text{for} \quad \underline{D}(y,\pm)\,\underline{A}(y) \qquad (m^{-1})$$

$$'\underline{D}(y,\pm)' \quad \text{for} \quad \text{diag}[D(y,\pm,1),\ldots,D(y,\pm,m)]$$

and

$$'\underline{A}(y)' \quad \text{for} \quad \underline{a}(y) - \underline{\hat{s}}(y) \qquad (m^{-1})$$

The derivation proceeds basically as in H.O., vol. V, p. 118, and now uses (12.28) as the required version of the divergence relation.

H. *Characteristic Equation for* $\underline{K}$

In the monochromatic theory a defining equation for the pair $K(y,\pm)$ in terms of $a(y,\pm)$ and $b(y,\pm)$ is given by (cf. H.O., vol. V, p. 123)

$$\frac{b(y,-)}{K(y,-) - a(y,-)} - \frac{b(y,+)}{K(y,+) + a(y,+)} = 1 \tag{12.34}$$

Clearing of fractions, we obtain a quadratic equation in K that yields $K(y,\pm)$ as roots. When a,b are independent of depth, these roots coincide with the classical functions $k_{\pm}$ of the monochromatic two-flow model for the light field (cf. H.O., vol. V, eq. (12), p. 31). In the present heterochromatic version of the two-flow theory, (12.34) generalizes to

$$\boxed{[\underline{K}(y,-) + \underline{\tau}(y,-)]\,\underline{\rho}^{-1}(y,+)[\underline{K}(y,+) - \underline{\tau}(y,+)] = -\underline{\rho}(y,-)} \tag{12.35}$$

which follows by eliminating $\underline{R}(y,-)$ from (12.26). Replacing $\underline{K}(y,\pm)$ in (12.35) by a single unknown m×m matrix $\underline{K}$ yields a quadratic matrix equation in $\underline{K}$ whose 2m roots (the *characteristic roots*) are the present generalizations of $k_{\pm}$ in the monochromatic theory (the case for m = 1).

PART III. DIRECT SOLUTION OF IRRADIANCE MODEL

13. EIGENMATRIX SOLUTION OF THE FLUORESCENT LIGHT FIELD MODEL

We now begin our constructions of the direct solution procedure of the fluorescent irradiance model. The characteristic equation (12.35) suggests an interesting path to the solution. This path is by way of a generalization of the classical monochromatic two-flow model k-functions, namely k_+ and k_-, to the fluorescent case (i.e., $m > 1$). With such a generalization there is the possibility of representing the λ-averaged irradiances $H(y,\pm,j)$ in (11.16) as linear combinations of simple exponential functions. This may be compared to the corresponding situation in H.O., vol. V, Eq. (9), p. 31, for the monochromatic case. If this approach is followed, it leads to a matrix-theoretic solution formalism (the fundamental-solution procedure) for (11.19). We shall now explore this idea just far enough to see that the generalization of $k_{\pm}$ to the heterochromatic case is possible and potentially thoroughgoing. In this way we can cap the list of analogies between the monochromatic and heterochromatic models developed in section 12 by presenting a very useful matrix formulation of the solution of the heterochromatic case. This formulation will be called the *eigenmatrix solution*.

The special eigen-representation we discuss here can only exist in a homogeneous medium or within homogeneous layers comprising a stratified medium. Therefore, for the remainder of this section we assume that the local transmittance and reflectance matrices $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$ are independent of y, $x \le y \le z$; and we also set $\underline{h}_e(y,\pm) = \underline{0}$ for all y, $x \le y \le z$ in an optical medium $X(x,z)$ until further notice. A generalization of the present theory to continuously stratified media with sources will be made in section 14.

We begin by writing the coupled pair of equations (11.19) as a single system. Thus write

$$'\underline{H}(y)' \text{ for } [\underline{H}(y,+),\underline{H}(y,-)] \qquad (1 \times 2m) \tag{13.1}$$

and

$$'\underline{K}' \text{ for } \begin{bmatrix} -\underline{\tau}(+) & \underline{\rho}(+) \\ \\ -\underline{\rho}(-) & \underline{\tau}(-) \end{bmatrix} \qquad (2m \times 2m) \tag{13.2}$$

Then (11.19) becomes

$$\frac{d}{dy}\,\underline{H}(y) = \underline{H}(y)\,\underline{K} \tag{13.3}$$

We wish to find an 'upward' and a 'downward' pair of 1×m irradiance vectors $\underline{\tilde{H}}(y,+)$ and $\underline{\tilde{H}}(y,-)$ ($W \cdot m^{-2} \cdot nm^{-1}$) analogous to $\underline{H}(y,+)$ and $\underline{H}(y,-)$, but with the unique property that

$$\frac{d}{dy}\,\underline{\tilde{H}}(y,\pm) = \underline{\tilde{H}}(y,\pm)\,\underline{k}_{\pm} \tag{13.4}$$

where we write

$$'\underline{k}_{\pm}' \quad \text{for} \quad \mathrm{diag}[k_{\pm}(1),\ldots,k_{\pm}(m)] \qquad (m^{-1}) \tag{13.5}$$

In other words, if we define the m components of $\underline{\tilde{H}}(y,\pm)$ in context by

$$\underline{\tilde{H}}(y,\pm) \equiv [\tilde{H}(y,\pm,1),\ldots,\tilde{H}(y,\pm,m)] \qquad (W \cdot m^{-2} \cdot nm^{-1}) \tag{13.6}$$

then (13.4) states that for j = 1,...,m,

$$\frac{d}{dy}\,\tilde{H}(y,\pm,j) = \tilde{H}(y,\pm,j)\,k_{\pm}(j) \tag{13.7}$$

over the depth range y, $x \le y \le z$, in the homogeneous medium X(a,b). Therefore in the search for $\underline{\tilde{H}}(y,\pm)$ we must also find the 2m parameters $k_{\pm}(j)$, $j = 1,\ldots,m$. The beauty of (13.7) lies in its complete decoupling of the two irradiances $\tilde{H}(y,\pm,i)$ and $\tilde{H}(y,\pm,j)$ when $i \ne j$. The connection between $\underline{\tilde{H}}(y,\pm)$ and $\underline{H}(y,\pm)$ is assumed to be a linear connection, summarized by some as yet unknown $2m \times 2m$ dimensionless matrix $\underline{E}$, such that for all $y \in [x,z]$,

$$\underline{\tilde{H}}(y) \equiv \underline{H}(y)\ \underline{E} \tag{13.8}$$

where we have written

$$'\underline{\tilde{H}}(y)' \quad \text{for} \quad [\underline{\tilde{H}}(y,+),\underline{\tilde{H}}(y,-)] \qquad (1 \times 2m) \tag{13.9}$$

We call $\underline{\tilde{H}}(y)$ and its subvectors $\underline{\tilde{H}}(y,\pm)$ the *eigen irradiances*, to keep them conceptually distinct from the *physical irradiances* $\underline{H}(y) = [\underline{H}(y,+),\underline{H}(y,-)]$. From (13.7) we see that in homogeneous media the eigen irradiances decay in precisely an exponential manner.

The conditions (13.4) and (13.8) are sufficient to fix the forms of the structures $\underline{E}$ and $\underline{k}_{\pm}$. For, on writing (13.4) as

$$\frac{d}{dy}\ \underline{\tilde{H}}(y) = \underline{\tilde{H}}(y)\ \underline{k} \tag{13.10}$$

where we have set

$$'\underline{k}' \text{ for } \mathrm{diag}[\underline{k}_+,\underline{k}_-] \qquad (2m \times 2m) \tag{13.11}$$

and using (13.8), we see that (13.10) becomes

$$\frac{d}{dy}\, \underline{H}(y)\, \underline{\mathit{E}} = \underline{H}(y)\, \underline{\mathit{E}}\, \underline{k} \tag{13.12}$$

By (13.3) this reduces to:

$$\underline{H}(y)\, \underline{\mathit{K}}\, \underline{\mathit{E}} = \underline{H}(y)\, \underline{\mathit{E}}\, \underline{k} \tag{13.13}$$

This relation must hold for a light field $\underline{H}(y)$ whose components $H(y,\pm,j)$, $j = 1,\ldots,m$ at each depth y can take on arbitrary values. These arbitrary values are taken while holding fixed the distribution factors $D(y,\pm,j)$, $j = 1,\ldots,m$, at y, and also the inherent optical properties throughout the present homogeneous optical medium $X(x,z)$ (cf. (11.17), (11.18)). Hence on these plausible physical grounds, (13.13) holds if and only if

$$\boxed{\underline{\mathit{K}}\, \underline{\mathit{E}} = \underline{\mathit{E}}\, \underline{k}} \qquad\qquad (2m \times 2m) \tag{13.14}$$

This represents a standard eigenvector/eigenvalue problem wherein the 2m columns of $\underline{\mathit{E}}$ are the required eigenvectors of $\underline{\mathit{K}}$ and the 2m numbers along the $2m \times 2m$ diagonal matrix $\underline{k}$ are the associated eigenvalues of $\underline{\mathit{K}}$. The entries of $\underline{E}$ are dimensionless and are fixed to within a common factor.

Generally in real media we expect (and therefore will assume here that) the eigenvalues of $\underline{\mathit{K}}$ are real and distinct.* Moreover, we expect (and assume)

* This expectation arises from the fact that real media at each wavelength λ exhibit absorption. This simple fact produces distinct k_+ and k_- in the monochromatic case. See H.O., vol. V, p. 32.

that half of the eigenvalues are positive and half are negative.* In this way the eigenvalues can be paired positive to negative, just as k_+ and k_- were paired in the monochromatic theory. It can be shown, for example, that if $\underline{\tau}(+) = \underline{\tau}(-)$ and $\underline{\rho}(+) = \underline{\rho}(-)$, then the 2m eigenvalues $k_{\underline{+}}(j)$, $j = 1,\ldots,m$ form m matched pairs where $k_-(j) = -k_+(j)$, $j = 1,\ldots,m$. On the principle of continuity of the eigenvalues of $\underline{K}$ with respect to continuous variation of the elements of $\underline{K}$, we see that it is plausible to expect that in real media we can pair a negative eigenvalue $k_-(j)$ to each positive eigenvalue $k_+(j)$, $j = 1,\ldots,m$. We can therefore arrange the distinct 2m values $k_{\underline{+}}(j)$ in ascending order and relable them in the following manner:

$$k_-(m) < \cdots < k_-(1) < 0 < k_+(1) < \cdots < k_+(m) \tag{13.15}$$

Once this is done, the associated eigenvectors of $\underline{K}$ are uniquely fixed to within a scalar factor. Of course, in various theoretical examples, in order to push the theory to its limits of physicality, we should be prepared to examine pathological departures from the above expected properties of the $k_{\underline{+}}(j)$, $j = 1,\ldots,m$. We will not explore such matters in the present introductory study of the Eigenmatrix method.

Let us define the 2m eigenvectors of $\underline{K}$ in context by writing

$$'\underline{E}' \text{ for } [\underline{e}_+(1) \cdots \underline{e}_+(m) \,\vdots\, \underline{e}_-(1) \cdots \underline{e}_-(m)] \qquad (2m \times 2m) \tag{13.16}$$

and in turn the components of and two m×1 subvectors of $\underline{e}_{\underline{+}}(j)$ are also defined in context

* This fact depends on our partition of the light field into upward and downward streams, resulting in the factors −1 in the left column of (13.2). These factors give rise to positive-negative eigenvalue pairs.

$$\underline{e}_+(j) \equiv \begin{bmatrix} e_{++}(1,j) \\ \vdots \\ e_{++}(m,j) \\ e_{-+}(1,j) \\ \vdots \\ e_{-+}(m,j) \end{bmatrix} \equiv \begin{bmatrix} \underline{e}_{++}(j) \\ \underline{e}_{-+}(j) \end{bmatrix}, \quad \underline{e}_-(j) \equiv \begin{bmatrix} e_{+-}(1,j) \\ \vdots \\ e_{+-}(m,j) \\ e_{--}(1,j) \\ \vdots \\ e_{--}(m,j) \end{bmatrix} \equiv \begin{bmatrix} \underline{e}_{+-}(j) \\ \underline{e}_{--}(j) \end{bmatrix} \tag{13.17}$$

With this notation we may rewrite (13.14) as

$$\begin{aligned} &\underline{K}[\underline{e}_+(1) \cdots \underline{e}_+(m) \vdots (\underline{e}_-(1) \cdots \underline{e}_-(m)] \\ &= [\underline{e}_+(1) \cdots \underline{e}_+(m) \vdots \underline{e}_-(1) \cdots \underline{e}_-(m)] \begin{bmatrix} \underline{k}_+ & \underline{0} \\ \underline{0} & \underline{k}_- \end{bmatrix} \\ &= [\underline{e}_+(1)\, \underline{k}_+ \cdots \underline{e}_+(m)\, \underline{k}_- \vdots \underline{e}_-(1)\, \underline{k}_- \cdots \underline{e}_-(m)\, \underline{k}_-] \end{aligned}$$

From this we read off, for $j = 1,\ldots,m$,

$$\underline{K}\; \underline{e}_\pm(j) = \underline{e}_\pm(j)\; k_\pm(j) \tag{13.18}$$

We conclude that $\underline{e}_\pm(j)$ is the eigenvector associated to $k_\pm(j)$, for each $j = 1,\ldots,m$, and where the $k_\pm(j)$ are ordered as in (13.15). This association can be used to uniquely name the eigenvectors in the solution subroutine for (13.14) and pair them to their respective eigenvalues $k_\pm(j)$. It should be noted that the eigenvectors generated by a subroutine are generally fixed in value except for a sign (±). We can agree on removing this ambiguity in the vectors $\underline{e}_\pm(j)$ by making their first components $e_{+\pm}(1,j)$ positive, $j = 1,\ldots,m$.

Finally, it will be useful to partition the $2m \times 2m$ matrix $\underline{E}$ into four $m \times m$ submatrices which we define in context by

$$\underline{E} \equiv \begin{bmatrix} \underline{E}_{++} & \underline{E}_{+-} \\ \underline{E}_{-+} & \underline{E}_{--} \end{bmatrix} \qquad (2m \times 2m) \tag{13.19}$$

Thus the element in the ith row and jth column of $\underline{E}_{++}$, for example, is $e_{++}(i,j)$, for $i,j = 1,\ldots,m$, as defined in (13.17). The inverse $\underline{E}^{-1}$ of $\underline{E}$ is presumed to exist and we shall write

$$'\underline{F}' \text{ for } \underline{E}^{-1} \equiv \begin{bmatrix} \underline{F}_{++} & \underline{F}_{+-} \\ \underline{F}_{-+} & \underline{F}_{--} \end{bmatrix} \tag{13.20}$$

where the four $m \times m$ submatrices $\underline{F}_{++}$, etc. are defined in context. The $m \times m$ block submatrices $\underline{F}_{++},\ldots,\underline{F}_{--}$ of $\underline{F}$ can be evaluated directly in terms of the block submatrices $\underline{E}_{++},\ldots,\underline{E}_{--}$ of $\underline{E}$ using a straight-forward technique. [See, e.g., (15.11)-(15.14), below.] Analogously to (13.16), (13.17), we define $\underline{F}$'s 2m column vectors in context by writing

$$'\underline{F}' \text{ for } [\underline{f}_{+}(1) \cdots \underline{f}_{+}(m) \,\vdots\, \underline{f}_{-}(1) \cdots \underline{f}_{-}(m)] \qquad (2m \times 2m)$$

and for $j = 1,\ldots,m$:

$$\underline{f}_{+}(j) \equiv \begin{bmatrix} f_{++}(1,j) \\ \vdots \\ f_{++}(m,j) \\ f_{-+}(1,j) \\ \vdots \\ f_{-+}(m,j) \end{bmatrix} \equiv \begin{bmatrix} \underline{f}_{++}(j) \\ \underline{f}_{-+}(j) \end{bmatrix}, \quad \underline{f}_{-}(j) \equiv \begin{bmatrix} f_{+-}(1,j) \\ \vdots \\ f_{+-}(m,j) \\ f_{--}(1,j) \\ \vdots \\ f_{--}(m,j) \end{bmatrix} \equiv \begin{bmatrix} \underline{f}_{+-}(j) \\ \underline{f}_{--}(j) \end{bmatrix} \tag{13.21}$$

Let us now assume that $\underline{E}$ and $\underline{k}$ have been numerically determined, and that $\underline{F}$ ($\equiv \underline{E}^{-1}$) exists and has also been evaluated. Then we have from an integration of (13.7) that

$$\tilde{H}(y,\pm,j) = \tilde{H}(u,\pm,j) \exp\{k_{\underline{\pm}}(j)(y-u)\} \tag{13.22}$$

for u, y ε [x,z] and j = 1,...,m. Hence knowing $\tilde{H}(u',\pm,j)$ at any level u, $x \le u \le z$, allows us via (13.22) to find $\tilde{H}(y,\pm,j)$ at all other levels y, $x \le y \le z$. From (13.8) and the notation of (13.19) we have in particular

$$\underline{\tilde{H}}(y,\pm) = \underline{H}(y,+)\ \underline{E}_{\underline{+\pm}} + \underline{H}(y,-)\ \underline{E}_{\underline{-\pm}} \tag{13.23}$$

or in more detail, for each j = 1,...,m,

$$\tilde{H}(y,\pm,j) = \sum_{i=1}^{m} H(y,+,i)\ e_{\underline{+\pm}}(i,j) + \sum_{i=1}^{m} H(y,-,i)\ e_{\underline{-\pm}}(i,j) \tag{13.24}$$

Thus knowing $H(y,\pm,i)$ at level y, we can find $\tilde{H}(y,\pm,j)$ at the same level y. Further, from (13.8) and knowledge of $\underline{F}$ (cf. (13.20)) we can find the physical irradiances $\underline{H}(y)$ by

$$\underline{H}(y) = \underline{\tilde{H}}(y)\ \underline{F} \tag{13.25}$$

From this, by (13.20),

$$\underline{H}(y,\pm) = \underline{\tilde{H}}(y,+)\ \underline{F}_{+\pm} + \underline{\tilde{H}}(y,-)\ \underline{F}_{-\pm} \tag{13.26}$$

In more detail, with (13.21), for each $j = 1,\ldots,m$

$$H(y,\pm,j) = \sum_{i=1}^{m} \tilde{H}(y,+,i)\ f_{+\pm}(i,j)$$

$$+ \sum_{j=1}^{m} \tilde{H}(y,-,i)\ f_{-\pm}(i,j) \tag{13.27}$$

Using (13.22) in this with u now chosen to be, say, x, we have

$$\boxed{\begin{aligned} H(y,\pm,j) = &\sum_{i=1}^{m} \tilde{H}(x,+,i)\ \exp\{k_+(i)(y-x)\}\ f_{+\pm}(i,j) \\ &+ \sum_{i=1}^{m} \tilde{H}(x,-,i)\ \exp\{k_-(i)(y-x)\}\ f_{-\pm}(i,j) \\ &x \le y \le z \\ &j = 1,\ldots,m \end{aligned}} \tag{13.28}$$

This is a representation of $H(y,\pm,j)$ as a linear superposition of simple exponentials that grow (+) or decay (−) with depth y increasing. We can write this solution completely in terms of physical irradiances $\underline{H}(x,\pm,i)$ by replacing the eigen irradiances $\tilde{H}(x,\pm,i)$ in (13.28) using (13.24) with y set to x. The resultant representation takes the form

$$\boxed{\begin{aligned} H(y,\pm,j) = &\sum_{i=1}^{m} H(x,+,i)\ m_{+\pm}(x,y|i,j) \\ &+ \sum_{i=1}^{m} H(x,-,i)\ m_{-\pm}(x,y|i,j) \end{aligned}} \tag{13.29}$$

where we have written for all $y \in [x,z]$ and $i,j = 1,\ldots,m$,

$$\text{'}m_{+\pm}(x,y|i,j)\text{'} \quad \text{for} \quad \sum_{\ell=1}^{m} [e_{++}(i,\ell)\ \exp\{k_+(\ell)(y-x)\}\ f_{+\pm}(\ell,j)$$
$$+ e_{+-}(i,\ell)\ \exp\{k_-(\ell)(y-x)\}\ f_{-\pm}(\ell,j)] \tag{13.30a}$$

and

$$\text{'}m_{-\pm}(x,y|i,j) \quad \text{for} \quad \sum_{\ell=1}^{m} [e_{-+}(i,\ell)\ \exp\{k_+(\ell)(y-x)\}\ f_{+\pm}(\ell,j)$$
$$+ e_{--}(i,\ell)\ \exp\{k_-(\ell)(y-x)\}\ f_{-\pm}(\ell,j)] \tag{13.30b}$$

Clearly the $m_{+\pm}$ and $m_{-\pm}$ quantities are dimensionless.

This form of the solution of (11.19) is readily evaluated numerically. To prepare for such solution procedures and their further applications, we elevate (13.29) to the matrix level. For this we need only write (13.22) as

$$\underline{\tilde{H}}(y,\pm) = \underline{\tilde{H}}(x,\pm)\ \exp\{\underline{k}_+(y-x)\} \tag{13.31}$$

where we have used (13.5) and (13.6). Moreover, on recalling (13.9) and (13.11), this may be written even more compactly, as

$$\underline{\tilde{H}}(y) = \underline{\tilde{H}}(x)\ \exp\{\underline{k}(y-x)\} \tag{13.32}$$

Then by (13.8) applied twice this becomes

$$\underline{H}(y)\ \underline{\mathcal{E}} = \underline{H}(x)\ \underline{\mathcal{E}}\ \exp\{\underline{k}(y-x)\}$$

whence, on using (13.20), we find the basic *mapping property* of $\underline{\mathcal{M}}(x,y)$:

$$\boxed{\begin{aligned} \underline{H}(y) &= \underline{H}(x)\ \underline{E}\ \exp\{\underline{k}(y-x)\}\ \underline{F} \\ &\equiv \underline{H}(x)\ \underline{M}(x,y) \end{aligned}} \tag{13.33}$$

which holds for all $y \in [x,z]$. We have defined in context the $2m \times 2m$ matrix $\underline{M}(x,y)$. Equation (13.33) is therefore the desired matrix version of (13.29), as may be verified by performing the indicated operations on $\underline{E}$, $\underline{k}$, and $\underline{F}$. Observe in particular that the matrix $\underline{M}(x,y)$ has the form

$$\underline{M}(x,y) = \begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix} \qquad (2m \times 2m) \tag{13.33a}$$

and where the (i,j) element of the $m \times m$ matrix $\underline{M}_{+\pm}(x,y)$ is $m_{+\pm}(x,y|i,j)$ in (13.30a) and the (i,j) element of the $m \times m$ matrix $\underline{M}_{-\pm}(x,y)$ is $m_{-\pm}(x,y|i,j)$ in (13.30b); $i,j = 1,\ldots,m$. Moreover, since $\underline{k}$ is a $2m \times 2m$ diagonal matrix, $\exp\{\underline{k}(y-x)\}$ is also a $2m \times 2m$ diagonal matrix of the form:

$$\exp\{\underline{k}(y-x)\} = \begin{bmatrix} e^{k_+(1)(y-x)} & & & & & 0 \\ & \ddots & & & & \\ & & e^{k_+(m)(y-x)} & & & \\ & & & e^{k_-(1)(y-x)} & & \\ & 0 & & & \ddots & \\ & & & & & e^{k_-(m)(y-x)} \end{bmatrix} \tag{13.34}$$

That (13.33) is actually a solution of the homogeneous version of (11.19a) may be verified as follows. Using (13.3), we differentiate each side of (13.33) and make some rearrangements:

$$
\begin{aligned}
\frac{d}{dy}\,\underline{\mathrm{H}}(y) &= \underline{\mathrm{H}}(x)\,\frac{d}{dy}\,\underline{M}(x,y) \\
&= \underline{\mathrm{H}}(x)\;\underline{E}\,\frac{d}{dy}\,\exp\{\underline{\mathrm{k}}(y-x)\}\;\underline{F} && \text{(a)} \\
&= \underline{\mathrm{H}}(x)\;\underline{E}\;\underline{\mathrm{k}}\;\exp\{\underline{\mathrm{k}}(y-x)\}\;\underline{F} && \text{(b)} \\
&= \underline{\mathrm{H}}(x)\;\underline{E}\;\exp\{\underline{\mathrm{k}}(y-x)\}\;\underline{\mathrm{k}}\;\underline{F} && \text{(c)} \\
&= \underline{\mathrm{H}}(x)\;\underline{E}\;\exp\{\underline{\mathrm{k}}(y-x)\}(\underline{F}\;\underline{E})\;\underline{\mathrm{k}}\;\underline{F} && \text{(d)} \\
&= \underline{\mathrm{H}}(x)\;[\underline{E}\;\exp\{\underline{\mathrm{k}}(y-x)\}\;\underline{F}](\underline{E}\;\underline{\mathrm{k}})\;\underline{F} && \text{(e)} \\
&= \underline{\mathrm{H}}(y)\;\underline{K}\;\underline{E}\;\underline{F} && \text{(f)} \\
&= \underline{\mathrm{H}}(y)\;\underline{K} && (13.35)
\end{aligned}
$$

This agrees with (13.3). Hence $\underline{\mathrm{H}}(y)$, as represented in (13.33), is a solution of the basic differential equations for the source-free fluorescent irradiance model, in the case of an homogeneous medium X(a,b).

It is of interest, particularly for theorists coming on this set of ideas for the first time, to discern the reason for the validity of each step in (13.35). For example, in going from (a) to (b) we use the reproducibility of the exponential function under differentiation; in going from (b) to (c) we use the diagonal structures of $\underline{\mathrm{k}}$ and $\exp\{\underline{\mathrm{k}}(y-x)\}$ and hence their commutativity; in going from (e) to (f) we use the mapping property (13.33) and the eigenvector equation (13.14).

The preceding proof may be summarized in terms of the $2m \times 2m$ $\underline{M}$-matrix as follows:

$$
\boxed{\frac{d}{dy}\,\underline{M}(x,y) = \underline{M}(x,y)\;\underline{K}} \qquad (13.36)
$$

where

$$\underline{M}(x,y) \equiv \begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix}$$

This matrix and its four m×m submatrices defined in context, with entries as in (13.30), are of fundamental importance to the present theory, as we shall see when we study it further, below. For now we observe the following additional properties of $\underline{M}$ that are immediately verified using the representation

$$\boxed{\underline{M}(u,v) \equiv \underline{E}\, \exp\{\underline{k}(v-u)\}\, \underline{F}\ , \qquad u,v \in [x,y]} \tag{13.37}$$

Thus, we have the

closure (or *propagation* or *group*) *property*:

$$\text{for all } u,v,w \in [x,z], \quad \underline{M}(u,v)\,\underline{M}(v,w) = \underline{M}(u,w) \tag{13.38}$$

identity property:

$$\text{for all } u \in [x,z], \quad \underline{M}(u,u) = \underline{I} \tag{13.39}$$

inverse property:

$$\text{for all } u,v \in [x,z], \quad \underline{M}^{-1}(u,v) = \underline{M}(v,u) \tag{13.40}$$

where $\underline{I}$ in (13.39) is the 2m × 2m identity matrix.

In other words, the set $\{\underline{M}(u,v):\ u,v \in [x,z]\}$ of 2m × 2m matrices under matrix multiplication forms a group which is isomorphic to the group of real numbers on the interval [0,z-x] under the operation of addition, modulo

$(z-x)$. From (13.33) and (13.38)-(13.40) we see how tightly knit the light field is, in the sense that knowing $\underline{H}(u)$ at any one level in $X(x,z)$, we can find $\underline{H}(v)$ at any other level: $\underline{H}(v) = \underline{H}(u)\,\underline{M}(u,v)$, $u,v \in [x,z]$.

We conclude this discussion of the fundamental matrix $\underline{M}(x,y)$ by giving a representation of its four submatrices in terms of the eigenstructures $\underline{E}$ and $\underline{k}$ of $\underline{K}$. Using (13.19) and (13.20) in (13.37) we find

$$\begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix} = \begin{bmatrix} \underline{E}_{++} & \underline{E}_{+-} \\ \underline{E}_{-+} & \underline{E}_{--} \end{bmatrix} \begin{bmatrix} e^{\underline{k}_+(y-x)} & \underline{0}_m \\ \underline{0}_m & e^{\underline{k}_-(y-x)} \end{bmatrix} \begin{bmatrix} \underline{F}_{++} & \underline{F}_{+-} \\ \underline{F}_{-+} & \underline{F}_{--} \end{bmatrix} \tag{13.41}$$

where $\underline{0}_m$ is an $m\times m$ zero matrix. From this, for $a \le x \le y \le b$,

$$\underline{M}_{++}(x,y) = \underline{E}_{++}e^{\underline{k}_+(y-x)}\,\underline{F}_{++} + \underline{E}_{+-}e^{\underline{k}_-(y-x)}\underline{F}_{-+} \tag{13.42a}$$

$$\underline{M}_{+-}(x,y) = \underline{E}_{++}e^{\underline{k}_+(y-x)}\,\underline{F}_{+-} + \underline{E}_{+-}e^{\underline{k}_-(y-x)}\underline{F}_{--} \tag{13.42b}$$

$$\underline{M}_{-+}(x,y) = \underline{E}_{-+}e^{\underline{k}_+(y-x)}\,\underline{F}_{++} + \underline{E}_{--}e^{\underline{k}_-(y-x)}\underline{F}_{-+} \tag{13.42c}$$

$$\underline{M}_{--}(x,y) = \underline{E}_{-+}e^{\underline{k}_+(y-x)}\,\underline{F}_{+-} + \underline{E}_{--}e^{\underline{k}_-(y-x)}\underline{F}_{--} \tag{13.42d}$$

Interchanging 'y' and 'x' in (13.42a-d) yields the representation for $\underline{M}(y,x)$, and hence, by (13.40), the representation for $M^{-1}(x,y)$.

14. FUNDAMENTAL-SOLUTION PROCEDURE

In this section we use the fundamental-solution procedure (as defined below) to develop the complete solution of the basic differential equation system (11.19), (11.20) for the upward and downward irradiance vectors $\underline{H}(y,\pm)$ in X(a,b). The integration problem for (11.19) is defined in detail in the discussion following the boundary conditions (11.20). Thus in X(a,b) it will be assumed that the incident irradiances $\underline{H}(a,-)$ and $\underline{H}(b,+)$ along with the true emission terms $\underline{h}_e(y,\pm)$, and also the local reflectances $\underline{\rho}(y,\pm)$ and local transmittances $\underline{\tau}(y,\pm)$ are known and continuous at all depths y in the depth interval [x,z] between the horizontal boundaries X(a,x), X(z,b) of the medium. The boundaries themselves each have a distinct quartet $r_{\pm},t_{\pm}$ of reflectances and transmittances specified such that the incident and response irradiances on each side of these boundaries are governed by (11.20). Our object here is to find useful algorithms that lead to numerical determinations of the internal irradiances $\underline{H}(y,\pm)$ for all y, $x \le y \le z$, and to $\underline{H}(a,+)$ and $\underline{H}(b,-)$, the response irradiances emerging from the boundaries. The internal response irradiances $\underline{H}(y,\pm)$ are physically generated by the incident boundary irradiances $\underline{H}(a,-)$ and $\underline{H}(b,+)$ along with the true emission sources $\underline{h}_e(y,\pm)$ within X(x,z).

The motivation for the present solution procedure is given in section 13: the reader may wish to consult that discussion before proceeding; we will initially draw freely from it in what follows.

A. *Fundamental Solution*

We begin by writing the source-free form of (11.19) in the matrix form (13.3), but now with full depth variation possible in $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$ and $\underline{h}_e(y,\pm)$. Thus we now have

$$\frac{d}{dy}\,\underline{H}(y) = \underline{H}(y)\,\underline{K}(y) \tag{14.1}$$

where we write

$$'\underline{K}(y)' \text{ for } \begin{bmatrix} -\underline{\tau}(y,+) & \underline{\rho}(y,+) \\ -\rho(y,-) & \underline{\tau}(y,-) \end{bmatrix} \qquad (m^{-1}) \tag{14.2}$$

for all depths y ε [x,y] (in meters). We wish to generalize the construction of the fundamental matrix $\underline{M}(u,v)$ of (13.37) with the result to hold also in arbitrarily stratified media X(a,b). Thus on the basis of our preliminary work with (13.36), we postulate the existence of a 2m × 2m (dimensionless) matrix $\underline{M}(x,y)$:

$$\underline{M}(x,y) \equiv \begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix} \qquad (2m \times 2m) \tag{14.3}$$

with its four indicated m×m submatrices such that $\underline{M}(x,y)$ satisfies the differential equation*

$$\frac{d}{dy}\,\underline{M}(x,y) = \underline{M}(x,y)\,\underline{K}(y) \tag{14.4}$$

subject to the initial condition

---

* Our introduction to $\underline{M}(x,y)$ has been through the intuitively and physically motivated eigen irradiance solutions (13.22) of the basic irradiance equations (13.3). Another way of looking at (14.1) is geometrically, via vector space theory. It can be shown that the vector solutions $\underline{H}(y)$ of (14.1) generate a 2m-dimensional vector space for each y, x ≤ y ≤ z. The interesting and useful fact is that the 2m rows of $\underline{M}(x,y)$ for each y form a basis of that vector space. See Coddington and Levinson (1955, p. 68). Hence the general solution of (14.1) can be given as linear combinations of the rows of $M(x,y)$ at each y, x ≤ y ≤ z. This will be the algebraic way to view our goal (14.18), below.

$$\underline{M}(x,x) = \underline{I}_{2m} \qquad (2m \times 2m) \tag{14.5}$$

In view of (13.39), we require that the four submatrices of $\underline{M}(x,x)$ satisfy

$$\left.\begin{aligned} \underline{M}_{++}(x,x) &= \underline{I}_m & \underline{M}_{+-}(x,x) &= \underline{0}_m \\ \underline{M}_{-+}(x,x) &= \underline{0}_m & \underline{M}_{--}(x,x) &= \underline{I}_m \end{aligned}\right\} \tag{14.6}$$

where $\underline{I}_m$ and $\underline{0}_m$ are m×m identity and zero matrices, respectively. On the numerical level this means that for $i,j = 1,\ldots,m$,

$$\left.\begin{aligned} m_{++}(x,x|i,j) &= \delta_{ij} & m_{+-}(x,x|i,j) &= 0 \\ m_{-+}(x,x|i,j) &= 0 & m_{--}(x,x|i,j) &= \delta_{ij} \end{aligned}\right\} \tag{14.7}$$

where δ_{ij} is Kronecker's delta (0 if $i \neq j$; 1 if $i = j$).

Equation (14.4) harbors four matrix differential equations which may be defined by first writing (14.4) as

$$\frac{\partial}{\partial y}\begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix} = \begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix} \begin{bmatrix} -\underline{\tau}(y,+) & \underline{\rho}(y,+) \\ -\underline{\rho}(y,-) & \underline{\tau}(y,-) \end{bmatrix}$$

whence

$$\left.\begin{aligned} \mp\frac{\partial}{\partial y}\underline{M}_{+\pm}(x,y) &= \underline{M}_{+\pm}(x,y)\,\underline{\tau}(y,\pm) + \underline{M}_{+\mp}(x,y)\,\underline{\rho}(y,\mp) \\ \mp\frac{\partial}{\partial y}\underline{M}_{-\pm}(x,y) &= \underline{M}_{-\pm}(x,y)\,\underline{\tau}(y,\pm) + \underline{M}_{-\mp}(x,y)\,\underline{\rho}(y,\mp) \end{aligned}\right\} \tag{14.8}$$

Let $m_{++}(x,y|i,j)$ be the ith row, jth column element of $\underline{M}_{++}(x,y)$, etc. Then for a numerical march of these equations from x to a variable location y in $[x,z]$, these are programmed in the following form, for $i,j = 1,\ldots,m$:

$$\boxed{\begin{aligned}\mp \frac{\partial}{\partial y}\, m_{+\pm}(x,y|i,j) &= \sum_{\ell=1}^{m} m_{+\pm}(x,y|i,\ell)\, \tau(y,\pm|\ell,j)\\ &\quad + \sum_{\ell=1}^{m} m_{+\mp}(x,y|i,\ell)\, \rho(y,\mp|\ell,j)\\ \mp \frac{\partial}{\partial y}\, m_{-\pm}(x,y|i,j) &= \sum_{\ell=1}^{m} m_{-\pm}(x,y|i,\ell)\, \tau(y,\pm|\ell,j)\\ &\quad + \sum_{\ell=1}^{m} m_{-\mp}(x,y|i,\ell)\, \rho(y,\mp|\ell,j)\end{aligned}} \tag{14.9}$$

with initial conditions given by (14.7). The matrix $\underline{M}(x,y)$ so found is the *fundamental (solution) matrix* of the system (14.1). The mathematical existence of such a solution is guaranteed by the assumed continuity of $\underline{\rho}(y,\pm)$ and $\underline{\tau}(y,\pm)$ with y. By integrating (14.9) also from y to z using $\underline{M}(y,y) = \underline{I}_m$, as an initial value, we arrive at the matrix $M(y,z)$. Multiplying $\underline{M}(x,y)$ and $\underline{M}(y,z)$ we find a matrix that would be obtained by integrating (14.9) from x to z with initial matrix $\underline{M}(x,x) = \underline{I}_m$. In this way we can numerically establish the general closure property (13.38). By integrating (14.9) backward from z to y, starting with $\underline{M}(z,z) = \underline{I}_m$, we find $\underline{M}(z,x)$, with the property $\underline{M}(x,z)\underline{M}(z,x) = \underline{I}_m$, thus establishing the general forms (13.39) and (13.40). Hence the group structure of the fundamental matrix $\underline{M}(x,y)$ found in the homogeneous setting (13.38)-(13.40) extends to generally stratified media X(a,b). The group structure is a compact mathematical statement of the uniqueness of $\underline{H}(v)$ given any initial value $\underline{H}(u)$ where u,v are in X[x,z]. Summarizing these findings, the fundamental matrix $\underline{M}(x,z)$ has the following properties. Let X[x,z] be a general layer of the hydrosol with continuous local properties $\underline{\rho}(y,\pm),\underline{\tau}(y,\pm)$. Let u,v,w be arbitrary depths in X[x,z]. Then we have the

$$\textit{group property} \quad \text{for all } u,v,w,\ \underline{M}(u,v)\ \underline{M}(v,w) = \underline{M}(u,w) \tag{14.10}$$

$$\textit{identity property} \quad \text{for all } u,\ \ \underline{M}(u,u) = \underline{I}_{2m} \tag{14.11}$$

$$\textit{inverse property} \quad \text{for all } u,v,\ \underline{M}^{-1}(u,v) = \underline{M}(v,u) \tag{14.12}$$

Knowing $\underline{H}(s)$ at any depth s, $s \in [x,z]$ we can construct the desired physical irradiance field at each depth $y \in [x,z]$ by writing

$$\text{'}\underline{H}(y)\text{' for } \underline{H}(s)\ \underline{M}(s,y) \tag{14.13}$$

This $\underline{H}(y)$ satisfies (14.1), which may be checked by directly differentiating it and using (14.4). Equation (14.13) is the *mapping property* of the fundamental matrix and is a direct generalization of (13.33).

B. *Source Solution*

We consider next the solution of (11.19) with emissive source term present:

$$\frac{d}{dy}\,\underline{H}(y) = \underline{H}(y)\ \underline{K}(y) + \underline{h}_e(y) \tag{14.14}$$

where we have written (cf. (11.19a)):

$$\text{'}\underline{h}_e(y)\text{'} \quad \text{for} \quad [-\underline{h}_e(y,+),\ \underline{h}_e(y,-)] \qquad (\mathrm{W\cdot m^{-3}\cdot nm^{-1}}) \tag{14.14a}$$

We will find the light field $\underline{H}_e(u,y)$ generated by the emissive source density terms $\underline{h}_e(s,\pm)$ distributed over all levels s in an arbitrary depth interval [u,y]. Thus imagine there are no irradiances incident on X(a,b) at levels a and b, i.e., $\underline{H}(a,-) = \underline{0}$ and $\underline{H}(b,+) = \underline{0}$. The only sources in X(x,z) are the $\underline{h}_e(s,\pm)$ representing true emission type irradiances generated per unit depth at each depth s. Hence over an infinitesimally shallow depth interval Δs, we have generated at depth s an initial spectral irradiance pair $\underline{h}_e(s,\pm)\Delta s$ in the upward (+) and downward (−) directions. These irradiances at level s initiate a contribution to the light field throughout X(x,z). In other words, this pair acts like the initial irradiance $\underline{H}(s)$ in (14.13), and it follows that the resultant irradiance field $\Delta\underline{H}_e(s,y)$ where we have written

$$`\Delta\underline{H}_e(s,y)' \quad \text{for} \quad \underline{h}_e(s)\ \underline{\mathcal{M}}(s,y)\ \Delta s \qquad (1 \times 2m) \qquad (\mathrm{W \cdot m^{-2} \cdot nm^{-1}}) \qquad (14.15)$$

satisfies (14.1) at all $y \neq s$. Adding up all these contributions for all depths s in the depth interval [u,y] we write, for arbitrary fixed u and variable y in [x,z],

$$`\underline{H}_e(u,y)' \quad \text{for} \quad \int_u^y \underline{h}_e(s)\ \underline{\mathcal{M}}(s,y)\ ds \qquad (1 \times 2m) \qquad (\mathrm{W \cdot m^{-2} \cdot nm^{-1}}) \qquad (14.16)$$

It is easy to verify that $\underline{H}_e(u,y)$, as a function of y, satisfies (14.14), i.e., that

$$\frac{\partial}{\partial y}\ \underline{H}_e(u,y) = \underline{H}_e(u,y)\ \underline{\mathcal{K}}(y) + \underline{h}_e(y) \qquad (14.16a)$$

This may be done by using Leibniz' rule for differentiating an integral with a variable integration range [u,y]. At a point in the derivation one will use

(14.11). Therefore we conclude that $\underline{H}_e(u,y)$ is a particular solution of (14.14), namely the *source solution based at* $u \in [x,z]$. This source solution consists of upward and downward emission-induced irradiances defined in context as

$$\underline{H}_e(u,y) \equiv [\underline{H}_e(u,y,+),\ \underline{H}_e(u,y,-)] \qquad (1 \times 2m)$$

$$(\mathrm{W \cdot m^{-2} \cdot nm^{-1}}) \qquad (14.17a)$$

where we define the components of $\underline{H}_e(u,y,\pm)$ in context:

$$\underline{H}_e(u,y,\pm) = [H_e(u,y,\pm,1),\ldots,H_e(u,y,\pm,m)] \qquad (1 \times m) \quad (14.17b)$$

C. *The Complete Solution*

A general solution of (14.14) based at $u \in [x,z]$ is therefore

$$\boxed{\begin{aligned} \underline{H}(y) &\equiv \underline{H}(u)\,\underline{\mathcal{M}}(u,y) + \int_u^y \underline{h}_e(s)\,\underline{\mathcal{M}}(s,y)\,ds \\ &\equiv \underline{H}(u)\,\underline{\mathcal{M}}(u,y) + \underline{H}_e(u,y) \\ & u,y \in [x,z]. \end{aligned}} \qquad \begin{aligned} &(1 \times 2m) \\ &(\mathrm{W \cdot m^{-2} \cdot nm^{-1}}) \end{aligned} \qquad (14.18)$$

This solution of (14.14) describes a light field in $X(x,y)$ for all y, $x \leq y \leq z$. This light field is generated by the externally incident irradiances $\underline{H}(a,-)$ and $\underline{H}(b,+)$ on the two boundaries and the emissive source density $\underline{h}_e(s)$ distributed throughout $X(x,z)$, $x \leq s \leq z$. In order to numerically evaluate $\underline{H}(y)$ we must therefore find how $\underline{H}(u)$ depends on $\underline{H}(a,-)$, $\underline{H}(b,+)$ and $\underline{h}_e(s)$, $x \leq s \leq z$. To find $\underline{H}(u)$ we must also explicitly use the information in the boundary conditions (11.20). Towards this end we write (14.18) in two-flow component form:

$$[\underline{H}(y,+),\underline{H}(y,-)] = [\underline{H}(u,+),\underline{H}(u,-)] \begin{bmatrix} \underline{M}_{++}(u,y) & \underline{M}_{+-}(u,y) \\ \underline{M}_{-+}(u,y) & \underline{M}_{--}(u,y) \end{bmatrix} + [\underline{H}_e(u,y,+),\underline{H}_e(u,y,-)] \tag{14.19}$$

That is,

$$\underline{H}(y,+) = \underline{H}(u,+)\ \underline{M}_{++}(u,y) + \underline{H}(u,-)\ \underline{M}_{-+}(u,y) + \underline{H}_e(u,y,+) \tag{14.20}$$

$$\underline{H}(y,-) = \underline{H}(u,+)\ \underline{M}_{+-}(u,y) + \underline{H}(u,-)\ \underline{M}_{--}(u,y) + \underline{H}_e(u,y,-) \tag{14.21}$$

In order to use the boundary conditions we must set u in (14.20) and (14.21) first equal to x and y equal to z. Then we begin again and set u equal to z and y to x. This results in the four equations

$$\underline{H}(z,+) = \underline{H}(x,+)\ \underline{M}_{++}(x,z) + \underline{H}(x,-)\ M_{-+}(x,z) + \underline{H}_e(x,z,+) \tag{14.22}$$

$$* \qquad \underline{H}(z,-) = \underline{H}(x,+)\ M_{+-}(x,z) + \underline{H}(x,-)\ M_{--}(x,z) + \underline{H}_e(x,z,-) \tag{14.23}$$

$$* \qquad \underline{H}(x,+) = \underline{H}(z,+)\ \underline{M}_{++}(z,x) = \underline{H}(z,-)\ \underline{M}_{-+}(z,x) + \underline{H}_e(z,x,+) \tag{14.24}$$

$$\underline{H}(x,-) = \underline{H}(z,+)\ \underline{M}_{+-}(z,x) = \underline{H}(z,-)\ \underline{M}_{--}(z,x) + \underline{H}_e(z,x,-) \tag{14.25}$$

For convenience we repeat the four boundary conditions (11.20) here:

$$\underline{H}(a,+) = \underline{H}(x,+)\ \underline{t}(x,a) + \underline{H}(a,-)\ \underline{r}(a,x) \tag{14.26}$$

$$* \qquad \underline{H}(x,-) = \underline{H}(x,+)\ \underline{r}(x,a) + \underline{H}(a,-)\ \underline{t}(a,x) \tag{14.27}$$

$$* \qquad \underline{H}(z,+) = \underline{H}(b,+)\ \underline{t}(b,z) + \underline{H}(z,-)\ \underline{r}(z,b) \tag{14.28}$$

$$\underline{H}(b,-) = \underline{H}(b,+)\ \underline{r}(b,z) + \underline{H}(z,-)\ \underline{t}(z,b) \tag{14.29}$$

The four basic unknowns in this set of equations are $\underline{H}(x,\pm)$ and $\underline{H}(z,\pm)$. These may be found by solving simultaneously the four starred equations indicated above. Using (14.27) in (14.23) we eliminate H(x,-) and using (14.28) in (14.24) we eliminate H(z,+). The results are

$$\underline{H}(z,-) = \underline{H}(x,+)\,\underline{A} + \underline{c} \tag{14.30}$$

$$\underline{H}(x,+) = \underline{H}(z,-)\,\underline{B} + \underline{d} \tag{14.31}$$

where $\underline{A}$, $\underline{B}$, are known matrices and $\underline{c}$, $\underline{d}$ are known vectors:

$$\underline{A} \equiv \underline{\mathcal{M}}_{+-}(x,z) + \underline{r}(x,a)\,\underline{\mathcal{M}}_{--}(x,z) \qquad (m \times m) \tag{14.32}$$

$$\underline{c} \equiv \underline{H}(a,-)\,\underline{t}(a,x)\,\underline{\mathcal{M}}_{--}(x,z) + \underline{H}_e(x,z,-) \qquad (1 \times m) \tag{14.33}$$

$$\underline{B} \equiv \underline{\mathcal{M}}_{-+}(z,x) + \underline{r}(z,b)\,\underline{\mathcal{M}}_{++}(z,x) \qquad (m \times m) \tag{14.34}$$

$$\underline{d} \equiv \underline{H}(b,+)\,\underline{t}(b,z)\,\underline{\mathcal{M}}_{++}(z,x) + \underline{H}_e(z,x,+) \qquad (1 \times m) \tag{14.35}$$

From (14.30), (14.31) we find

$$\boxed{\underline{H}(z,-) = [\underline{d}\,\underline{A} + \underline{c}][\underline{I} - \underline{B}\,\underline{A}]^{-1}} \tag{14.36}$$

and

$$\boxed{\underline{H}(x,+) = [\underline{c}\,\underline{B} + \underline{d}][\underline{I} - \underline{A}\,\underline{B}]^{-1}} \tag{14.37}$$

Returning to (14.27) we determine $\underline{H}(x,-)$ using $\underline{H}(x,+)$ just found and the given incident irradiance $\underline{H}(a,-)$. Returning to (14.28) we determine $\underline{H}(z,+)$ using $\underline{H}(z,-)$ just found and the incident irradiance $\underline{H}(b,+)$. In this way we may determine $\underline{H}(x,\pm)$ and $\underline{H}(z,\pm)$.

It remains only to find the emergent irradiances $\underline{H}(a,+)$ and $\underline{H}(b,-)$ at the upper and lower boundaries. These are obtained via (14.26) and (14.29); these are the remaining two as yet unused boundary conditions. This completes the solution. We may now use either $\underline{H}(x) = [\underline{H}(x,+), \underline{H}(x,-)]$ in (14.18) (on setting $u = x$), or we may use $\underline{H}(z) = [\underline{H}(z,+), \underline{H}(z,-)]$ in (14.18) (on setting $u = z$). In either case we can then find all other irradiance pairs $\underline{H}(y) = [\underline{H}(y,+), \underline{H}(y,-)]$ at any level y in $[x,z]$.

D. *Prelude to the Global Interaction Principles*

The reader may have observed in the preceding solution procedure that (14.22) and (14.25) were never used. This is because they are not independent of (14.23) and (14.24). We see here further evidence (recall the discussion of (13.38)-(13.40)) of the strong internal bond between the light fields at all pairs of levels in $[x,z]$ including the end levels x and z themselves in $X(x,z)$. A particular form in which this bond appears may be seen from (14.18) on first setting $u = x$ and $y = z$ to obtain

$$\underline{H}(z) = \underline{H}(x)\, \underline{\mathcal{M}}(x,z) + \underline{H}_e(x,z) \tag{14.38}$$

and then setting $u = z$ and $y = x$ in (14.18) to obtain

$$\underline{H}(x) = \underline{H}(z)\, \underline{\mathcal{M}}(z,x) + \underline{H}_e(z,x) \tag{14.39}$$

Substituting (14.39) into (14.38) and using (14.12), we find the first bond:

$$\boxed{\underline{H}_e(z,x) = -\underline{H}_e(x,z)\, \underline{\mathcal{M}}(z,x)} \tag{14.40a}$$

or equivalently

$$\boxed{\underline{H}_e(x,z) = -\underline{H}_e(z,x)\, \underline{\mathcal{M}}(x,z)} \tag{14.40b}$$

Using this in (14.39) we find the second bond:

$$\boxed{\begin{aligned} \underline{H}(x) &= [\underline{H}(z) - \underline{H}_e(x,z)]\ \underline{M}(z,x) \\ \underline{H}(z) &= [\underline{H}(x) - \underline{H}_e(z,x)]\ \underline{M}(x,z) \end{aligned}} \qquad \begin{aligned} &(14.41a) \\ &(14.41b) \end{aligned}$$

or equivalently

One may verify that the complete solution of (11.19), (11.20) can be obtained once again by repeating the preceding analysis beginning now with the unstarred equations in the octet (14.22)-(14.29). Therefore the octet (14.22)-(14.29) has two distinct but redundant basic quartets of equations from which the solution of (11.19), (11.20) may be deduced. This is attributable, as just seen, to the strong inner bond of the light field's parts in X(x,z).

As another example of the inner bond on the light field we consider the source-generated part $\underline{H}_e(x,z)$ of the field. By three applications of (14.18), we can write, for arbitrary u,v,w in [x,z]

$$\underline{H}(w) = \underline{H}(v)\ \underline{M}(v,w) + \underline{H}_e(v,w) \qquad (14.42a)$$

$$\underline{H}(v) = \underline{H}(u)\ \underline{M}(u,v) + \underline{H}_e(u,v) \qquad (14.42b)$$

and also

$$\underline{H}(w) = H(u)\ M(u,w) + \underline{H}_e(u,w) \qquad (14.42c)$$

Using (14.42b) in (14.42a), comparing the result with (14.42c), and letting $\underline{H}(u)$ and internal sources $\underline{h}_e(y,\pm)$, $x \le y \le z$ be arbitrary, we find (14.10) again

$$\underline{M}(u,w) = \underline{M}(u,v)\ \underline{M}(v,w)$$

as expected, and also something new:

$$\boxed{\underline{H}_e(u,w) = \underline{H}_e(u,v)\ \underline{M}(v,w) + \underline{H}_e(v,w)} \tag{14.43}$$

This relation is a form of addition law that shows how to construct $\underline{H}_e(u,w)$ over the interval [u,w] from knowledge of $\underline{H}_e(u,v)$ and $\underline{H}_e(v,w)$ over the subintervals [u,v] and [v,w] of [u,w]. Its transport version will be encountered in the union rules (18.7) and (18.10).

In sum, then, the strong inner bond between the light levels at two different depths in X(x,z), as used above to determine the complete solution of the local interaction principles (11.19) and their boundary conditions (11.20), is manifested for example in (14.40), (14.41), and (14.43). These in turn rest on the group properties (14.10)-(14.12) of the fundamental matrix $M(u,v)$. The fundamental matrix, which embodies these group properties, is designed to map the two flows $\underline{H}(u) = [\underline{H}(u,+), \underline{H}(u,-)]$ of the light field at level u into the two flows $\underline{H}(v) = [\underline{H}(v,+), \underline{H}(v,-)]$ at level v. In other words $\underline{M}(u,v)$ is the appropriate tool to solve the *one-level* or *one-point* boundary value problem, as it is called in modern differential equation theory.

Now recall that both sets of boundary conditions, namely (14.26), (14.27) over boundary X(a,x), and conditions (14.28), (14.29) over boundary X(z,b) had to be used to attain the complete solution of (11.19), (11.20) in the starred equation procedure on the octet (14.22)-(14.29); or alternatively, both sets of boundary conditions were used in the unstarred equation procedure on the octet (14.22)-(14.29). On reflection, this suggests the possibility that there may be a more natural solution procedure of the two-boundary radiative transfer problem (11.19), (11.20) on X(a,b), one that eliminates the observed redundancy between the starred and unstarred quartets of equations in (14.22)-(14.29). If one pursues this suggestion, one eventually arrives at the notion of the *global interaction principles*, which stand at the base of modern radiative transfer applications. These principles are the global counterparts to the local interaction principles (11.19) (for infinitesimal layers) and also on global counterparts to the boundary conditions (11.20) (for zero thickness surfaces). We turn next to the derivation of these global interaction principles. They will provide not only a more natural (i.e., a more physically intuitive) solution of (11.19), (11.20) than that provided by $\underline{M}(u,v)$, but they will also lead to numerical integration procedures of (11.19), (11.20) that are inherently more stable than those provided by (14.4), (14.5).

15. GLOBAL INTERACTION PRINCIPLES

The global interaction principles, to be developed below, represent the two-boundary radiative transfer problem (11.19), (11.20) in a direct and intuitive manner. They also lead, as we will see in section 16, to numerically stable integration procedures for (11.19). The numerical distinction between the fundamental-solution procedure of section 14 and the transport-solution procedure of section 16 is between a one-point and a two-point boundary value procedure, and also between an unstable and a stable numerical integration procedure for (11.19), respectively. Both procedures of course are mathematically equivalent and, for moderately deep media lead to the same numerical values of the light field $\underline{H}(y,\pm)$ in $X(a,b)$, given (11.19), (11.20). However, as we shall try to show, the transport procedure's equations are interpretable in terms of the reflection and transmission of radiant flux through and between the various contiguous sublayers of a stratified optical medium. Such high physical interpretability endows the transport solution procedure with heuristic advantages particularly in checking the validity of newly derived relations of the light field, and in suggesting further new ones. In our view, the fundamental solution of sections 13 and 14 serves two main purposes in radiative transfer theory of lakes and seas: (1) it provides us, as we have seen in section 13, with the beautiful algebraic structures of the eigenmatrix solution, and (2) it provides a rigorous mathematical foundation for the global interaction principles.* We now briefly outline this latter feature of the fundamental solution.

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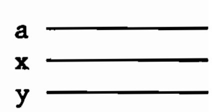
* At one time in the history of radiative transfer theory, the global interaction principles, in their early form, known as the *principles of invariance*, were invoked on a purely intuitive basis. This was perfectly acceptable, as they produced useful and correct solution procedures. Later, they were derived from a linear interaction principle (Preisendorfer, 1965); and still later, they were shown to come from the fundamental solution (H.O., vol. IV, sec. 7.4). See also (Preisendorfer, 1977, p. 33).

Throughout this section the setting is a plane parallel stratified lake or sea $X(a,b)$ with upper surface at level a and lower surface at b, and various level surfaces at depths x,y,z, etc., $a \le x \le y \le z \le b$. The upper boundary $X(a,x)$ and lower boundary $X(z,b)$ may be zero thickness surfaces (as in the case of the air-water surface or bottom surface of a lake or sea) or slabs of finite thickness with internal sources. In either case, as in sections 13 and 14, these boundary media will be postulated to have known reflectance and transmittance quartets $\underline{r}_{\pm}, \underline{t}_{\pm}$ and known source-generated irradiances $\underline{H}_{e\pm}$. Moreover, the internal (water body) region $X(x,z)$ of $X(a,b)$ has specified continuously varying local reflectance and transmittance functions $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$ and continuously varying emission density functions $\underline{h}_e(y,\pm)$, $x \le y \le z$.

A. *Fundamental to Transport Solution. Downward Case*

We consider first a submedium $X(a,y)$ of $X(a,b)$. We direct attention to the variable depth y below the upper boundary layer $X(a,x)$ (see the ideograph, below, for Eq. 15.1). It is assumed that $\underline{H}(a)$ along with $\underline{\mathcal{M}}(a,x)$ are known ($\underline{H}(a)$ follows, as shown, from the complete solution in section 14C. A method for calculating the matrix $\underline{\mathcal{M}}(a,x)$ explicitly from the hypothesized known quartet $\underline{r}(a,x)$, $\underline{r}(x,a)$, $\underline{t}(a,x)$, $\underline{t}(x,a)$ will automatically evolve from the present discussion; cf. par. B). Since we know $\underline{H}(a)$ and $\underline{\mathcal{M}}(a,x)$, we then know $\underline{H}(x) = \underline{H}(a)\,\underline{\mathcal{M}}(a,x)$ in order to start the integration. Then we can integrate (11.19) in the form



$$\frac{d}{dy}[\underline{H}(y,+), \underline{H}(y,-)] = [\underline{H}(y,+), \underline{H}(y,-)]\,\underline{\mathcal{K}}(y) + [-\underline{h}_e(y,+), h_e(y,-)] \tag{15.1}$$

from x to a lower, variable depth y in X[x,z]. By (14.18), we can write the general solution of (15.1) as

$$[\underline{H}(y,+),\ \underline{H}(y,-)] = [\underline{H}(x,+),\ \underline{H}(x,-)]\ \underline{\mathcal{M}}(x,y) + [\underline{H}_e(x,y,+),\ \underline{H}_e(x,y,-)]$$

$$x \le y \le z \tag{15.2}$$

Now, the *transport solution procedure* of (15.1) views the submedium X(x,y) as being irradiated by the downward irradiance $\underline{H}(x,-)$ at level x and by the upward irradiance $\underline{H}(y,+)$ at level y. These *incident* irradiances, along with the two internal source flows $\underline{h}_e(s,\pm)$, produce *response* irradiances $\underline{H}(x,+)$ and $\underline{H}(y,-)$ leaving X(x,y) at levels x and y. The primary feature of the transport solution of (15.1) is to supply a linear transformation between these incident and response irradiances at levels x and y. Towards this end we reassemble the irradiances in (15.2) into the equivalent form:

$$\underbrace{[\underline{H}(y,+),\ \underline{H}(y,-),}_{\substack{\text{fundamental}\\ \text{output}}}\ \underbrace{\underline{H}(x,+),\ \underline{H}(x,-)]}_{\substack{\text{fundamental}\\ \text{input}}} \begin{bmatrix} \underline{I}_{2m} \\ \\ -\underline{\mathcal{M}}(x,y) \end{bmatrix} = \underbrace{[\underline{H}_e(x,y,+),\ \underline{H}_e(x,y,-)]}_{\text{source-generated}} \tag{15.3}$$

where $\underline{I}_{2m}$ is a 2m × 2m matrix along with $\underline{\mathcal{M}}(x,y)$. Hence the matrix in the square brackets is a 4m × 2m matrix that maps on the 1 × 4m output-input vector on the left to yield the 1 × 2m source-generated vector on the right of (15.3). This formulation, and particularly that of (15.2), views [H(x,+), H(x,-)] as input irradiances while [H(y,+), H(y,-)] are output irradiances in the fundamental-solution context.

Now, we wish to rearrange the irradiances in the $1 \times 4m$ output-input vector of (15.3) so that the response irradiances $\underline{H}(x,+)$, $\underline{H}(y,-)$ of $X(x,y)$ are together and the incident irradiances $\underline{H}(x,-)$, $\underline{H}(y,+)$ are together; these are the *output* and *input irradiances*, respectively, of the transport formulation of the radiative transport process on the submedium $X[x,y]$ of $X[a,b]$. We can schematically link these two kinds of input and output vectors in the fundamental and the transport procedures as follows:

$$\underbrace{[\underline{H}(y,+),\underline{H}(y,-)}_{\substack{\text{fundamental}\\ \text{output}}},\underbrace{\underline{H}(x,+),\underline{H}(x,-)]}_{\substack{\text{fundamental}\\ \text{input}}}\ \underbrace{\begin{array}{c} \begin{array}{cc} \underline{C}_- \qquad & \qquad \underline{C}_+ \end{array} \\ \left[\begin{array}{cc|cc} \underline{O}_m & \underline{O}_m & \underline{I}_m & \underline{O}_m \\ \underline{O}_m & \underline{I}_m & \underline{O}_m & \underline{O}_m \\ \hline \underline{I}_m & \underline{O}_m & \underline{O}_m & \underline{O}_m \\ \underline{O}_m & \underline{O}_m & \underline{O}_m & \underline{I}_m \end{array}\right] \\ \begin{array}{cc} \underline{C}_+ \qquad & \qquad \underline{C}_- \end{array} \end{array}}_{\underline{Q}_-} = \underbrace{[\underline{H}(x,+),\underline{H}(y,-)}_{\substack{\text{transport}\\ \text{output}}},\underbrace{\underline{H}(y,+),\underline{H}(x,-)]}_{\substack{\text{transport}\\ \text{input}}}$$

vectors (fundamental solution) — vectors (transport solution)

(15.4)

Here $\underline{I}_m$ and $\underline{O}_m$ are $m \times m$ identity and zero matrices. Thus the $4m \times 4m$ matrix $\underline{Q}_-$ maps the fundamental input-output vectors into their transport input-output arrangement. The $2m \times 2m$ matrices of $\underline{Q}_-$ are defined in context in (15.4). It is readily checked that

$$\begin{bmatrix} \underline{C}_- & \underline{C}_+ \\ \underline{C}_+ & \underline{C}_- \end{bmatrix}^2 = \underline{Q}_-^2 = \underline{I}_{4m} \tag{15.5}$$

We now insert $\underline{Q}_-^2$ ($= \underline{I}_{4m}$) between the two factors on the left side of (15.3) to obtain

$$[\underline{H}(y,+),\underline{H}(y,-),\underline{H}(x,+),\underline{H}(x,-)]\ \underline{Q}_-^2 \begin{bmatrix} \underline{I}_{2m} \\ -M(x,y) \end{bmatrix} = [\underline{H}_e(x,y,+),\underline{H}_e(x,y,-)] \tag{15.6}$$

Then using the action of $\underline{Q}_-$ in (15.4) on each of the factors flanking $\underline{Q}_-^2$, this becomes

$$\underbrace{[\underline{H}(x,+),\underline{H}(y,-),}_{\substack{\text{transport}\\\text{output}}}\ \underbrace{\underline{H}(y,+),\underline{H}(x,-)]}_{\substack{\text{transport}\\\text{input}}} \begin{bmatrix} \underline{C}_- - \underline{C}_+\underline{M}(x,y) \\ \underline{C}_+ - \underline{C}_-\underline{M}(x,y) \end{bmatrix} = \underbrace{[\underline{H}_e(x,y,+),\underline{H}_e(x,y,-)]}_{\text{source generated}} \tag{15.7}$$

Then on further reduction this may be written

$$[\underline{H}(x,+),\underline{H}(y,-)] = [\underline{H}(y,+),\underline{H}(x,-)]\ [\underline{C}_-\underline{M}(x,y) - \underline{C}_+]\ [\underline{C}_- - \underline{C}_+\underline{M}(x,y)]^{-1}$$

$$+ [\underline{H}_e(x,y,+),\underline{H}_e(x,y,-)][\underline{C}_- - \underline{C}_+\underline{M}(x,y)]^{-1} \tag{15.8}$$

This is the desired reformulation of (15.2). Working out the indicated matrix operations on $\underline{M}(x,y)$ we find

$$[\underline{C}_- - \underline{C}_+\underline{M}(x,y)]^{-1} = \left\{ \begin{bmatrix} \underline{O}_m & \underline{O}_m \\ \underline{O}_m & \underline{I}_m \end{bmatrix} - \begin{bmatrix} \underline{I}_m & \underline{O}_m \\ \underline{O}_m & \underline{O}_m \end{bmatrix} \begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix} \right\}^{-1} \tag{15.9}$$

$$= \begin{bmatrix} -\underline{M}_{++}(x,y) & -\underline{M}_{+-}(x,y) \\ \underline{O}_m & \underline{I}_m \end{bmatrix}^{-1} \tag{15.10}$$

Now, to invert a $2m \times 2m$ block matrix of the form

$$\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{bmatrix} \qquad (2m \times 2m) \tag{15.11}$$

where $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ are given m×m matrices, there is a 2m × 2m block matrix

$$\begin{bmatrix} \underline{W} & \underline{X} \\ \underline{Y} & \underline{Z} \end{bmatrix} \qquad (2m \times 2m) \tag{15.12}$$

with m×m submatrices $\underline{W}$, $\underline{X}$, $\underline{Y}$, $\underline{Z}$ such that

$$\begin{bmatrix} \underline{A} & \underline{B} \\ \underline{C} & \underline{D} \end{bmatrix} \begin{bmatrix} \underline{W} & \underline{X} \\ \underline{Y} & \underline{Z} \end{bmatrix} = \begin{bmatrix} \underline{I}_m & \underline{0}_m \\ \underline{0}_m & \underline{I}_m \end{bmatrix} \tag{15.13}$$

where

$$\left.\begin{aligned} \underline{W} &= [\underline{A} - \underline{B}\,\underline{D}^{-1}\,\underline{C}]^{-1} & \underline{X} &= [\underline{C} - \underline{D}\,\underline{B}^{-1}\,\underline{A}]^{-1} \\ &= -\underline{C}^{-1}\,\underline{D}\,\underline{Y} & &= -\underline{A}^{-1}\,\underline{B}\,\underline{Z} \\ \underline{Y} &= [\underline{B} - \underline{A}\,\underline{C}^{-1}\,\underline{D}]^{-1} & \underline{Z} &= [\underline{D} - \underline{C}\,\underline{A}^{-1}\,\underline{B}]^{-1} \\ &= -\underline{D}^{-1}\,\underline{C}\,\underline{W} & &= -\underline{B}^{-1}\,\underline{A}\,\underline{X} \end{aligned}\right\} \tag{15.14}$$

The alternate expressions for $\underline{W}$, $\underline{X}$, $\underline{Y}$, $\underline{Z}$ may generally be used if various inverses of $\underline{A}$, $\underline{B}$, $\underline{C}$, $\underline{D}$ do not exist. For example in the expression for $\underline{X}$, if $\underline{B} = \underline{0}$, then the first form is inapplicable and the alternate gives $\underline{X} = \underline{0}$. In transport theory, the actual counterparts to $\underline{A}$ and $\underline{D}$ are transmittance-type matrices and therefore tend to have inverses $\underline{A}^{-1}$ and $\underline{D}^{-1}$. Hence we usually can employ the first forms of $\underline{W}$ and $\underline{Z}$, and the second forms of $\underline{X}$ and $\underline{Y}$. Applying (15.14) to (15.10), we set $\underline{A} = -\underline{M}_{++}(x,y)$, $\underline{B} = -\underline{M}_{+-}(x,y)$, $\underline{C} = \underline{0}_m$, $\underline{D} = \underline{I}_m$. Then $\underline{W} = -\underline{M}_{++}^{-1}(x,y)$, $\underline{X} = -\underline{M}_{++}^{-1}(x,y)\,\underline{M}_{+-}(x,y)$, $\underline{Y} = \underline{0}_m$, $\underline{Z} = \underline{I}_m$, so that

$$[\underline{C}_- - \underline{C}_+\underline{M}(x,y)]^{-1} = \begin{bmatrix} -\underline{M}_{++}^{-1}(x,y) & -\underline{M}_{++}^{-1}(x,y)\underline{M}_{+-}(x,y) \\ \underline{0}_m & \underline{I}_m \end{bmatrix} \tag{15.15}$$

Hence we arrive at

$$\underline{M}(x,y) \equiv \begin{bmatrix} \underline{T}(y,x) & \underline{R}(y,x) \\ \underline{R}(x,y) & \underline{T}(x,y) \end{bmatrix} \equiv [\underline{C}_{-}\underline{M}(x,y) - \underline{C}_{+}][\underline{C}_{-} - \underline{C}_{+}\underline{M}(x,y)]^{-1} \qquad (15.16)$$

$$= \left[\begin{array}{c:c} \underline{M}_{++}^{-1}(x,y) & \underline{M}_{++}^{-1}(x,y)\ \underline{M}_{+-}(x,y) \\ \hdashline -\underline{M}_{-+}(x,y)\ \underline{M}_{++}^{-1}(x,y) & \underline{M}_{--}(x,y) - \underline{M}_{-+}(x,y)\ \underline{M}_{++}^{-1}(x,y)\ \underline{M}_{+-}(x,y) \end{array}\right] \qquad (15.17)$$

In this way we define in context the four (dimensionless) transfer functions $T(x,y)$, $R(x,y)$, $R(y,x)$, and $T(x,y)$ shown in (15.16). $\underline{M}(x,y)$ is called the *transport matrix*. Further, evaluating the source-generated term in (15.8) we write in context

$$'[\underline{H}_\eta(y,x),\ \underline{H}_\eta(x,y)]' \quad \text{for} \qquad (W\cdot m^{-2}\cdot nm^{-1})$$
$$[-\underline{H}_e(x,y,+)\ \underline{M}_{++}^{-1}(x,y),\ \underline{H}_e(x,y,-) - \underline{H}_e(x,y,+)\ \underline{M}_{++}^{-1}(x,y)\ \underline{M}_{+-}(x,y)] \qquad (15.18)$$

Observe that $\underline{H}_\eta(x,x) = \underline{0}$.

Equation (15.8) then takes the desired transport form:

$$\boxed{\begin{array}{l} [\underline{H}(x,+),\underline{H}(y,-)] = [\underline{H}(y,+),\underline{H}(x,-)]\ \underline{M}(x,y) + [\underline{H}_\eta(y,x),\underline{H}_\eta(x,y)] \\ x \le y \end{array}} \qquad (15.19)$$

Equations (15.19) are the *downward global interaction principles* for the medium $X(x,y)$. Written out, they are

$$\boxed{\begin{array}{l} \underline{H}(x,+) = \underline{H}(y,+)\ \underline{T}(y,x) + \underline{H}(x,-)\ \underline{R}(x,y) + \underline{H}_\eta(y,x) \\ \underline{H}(y,-) = \underline{H}(y,+)\ \underline{R}(y,x) + \underline{H}(x,-)\ \underline{T}(x,y) + \underline{H}_\eta(x,y) \\ x \le y \end{array}} \qquad \begin{array}{l} a \ \rule{2cm}{0.4pt} \\ x \ \rule{2cm}{0.4pt} \\ y \ \rule{2cm}{0.4pt} \\ z \ \rule{2cm}{0.4pt} \\ b \ \rule{2cm}{0.4pt} \end{array} \tag{15.20}$$

The term 'downward' in the name of these principles is a mnemonic for the fact that x is fixed while y increases downward. This view will be useful when later we are deriving the differential equations for the quartet $\underline{R}(y,x)$, $\underline{T}(x,y)$, $\underline{T}(y,x)$, $\underline{R}(x,y)$ of matrices, and also the differential equations for $\underline{H}_\eta(y,x)$, $\underline{H}_\eta(x,y)$ shown in (15.20).

The physical interpretation of the equation for the response irradiance $\underline{H}(x,+)$ in (15.20) is readily made. $\underline{H}(x,+)$ is the sum of the downward incident $\underline{H}(x,-)$ being reflected upward from X(x,y) at level x and the upward incident $\underline{H}(y,+)$ being transmitted from y to x and, finally, the irradiance $\underline{H}_\eta(y,x)$ emerging upward at x generated by emission sources within X(x,y). A similar interpretation of $\underline{H}(y,-)$ can be made. Observe that as y approaches x, so that in the limit the depth of X(x,y) vanishes, then

$$\begin{array}{l} \lim\limits_{y \to x} \underline{R}(x,y) = \lim\limits_{y \to x} \underline{R}(y,x) = \underline{0}_m \\ \lim\limits_{y \to x} \underline{T}(x,y) = \lim\limits_{y \to x} \underline{T}(y,x) = \underline{I}_m \end{array} \qquad (m \times m) \tag{15.21}$$

and also

$$\lim_{y \to x} \underline{H}_\eta(y,x) = \lim_{y \to x} \underline{H}_\eta(x,y) = \underline{0} \qquad (1 \times m) \tag{15.22}$$

These properties follow at once from their various definitions in (15.17) and (15.18) and the assumed continuity of $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$, and $\underline{h}_e(y,\pm)$ in X(x,z), as used in the local interaction principles (11.19). If the $\underline{\rho}$ and $\underline{\tau}$ are not continuous, say at some depth u in [x,y], physically this means there can be a reflecting surface of zero thickness at u. Such a surface generally will have a nontrivial $\underline{r}_{\pm},\underline{t}_{\pm}$ quartet (i.e., $\underline{r}_{\pm} \neq \underline{0}$, $\underline{t}_{\pm} \neq \underline{I}_m$) of reflectance and transmittance matrices, analogous to the air-water surface. This possibility is readily handled, in solving for the light field throughout X(x,y), by means of the union rule, developed in section 18C, D, below. It is only when we are *integrating* the differential equations for $\underline{H}(y,\pm)$ or $\underline{\mathcal{M}}(x,u)$ over X(x,y) that we must have continuity of $\underline{\rho}(u,\pm)$ and $\underline{\tau}(u,\pm)$ throughout X(x,y). This continuity in particular insures that (15.21) holds, and continuity of $\underline{h}_e(u,\pm)$ allows (15.22) to hold.

B. *Transport to Fundamental Solution. Downward Case*

In the opening remarks of par. A it was observed that we needed the boundary matrix $\underline{\mathcal{M}}(a,x)$ in order to begin our analysis. It was observed that $\underline{\mathcal{M}}(a,x)$ could be derived from the $\underline{r}_{\pm},\underline{t}_{\pm}$ quartet of the air-water surface. This quartet forms a special case of the $\underline{M}$-matrix in (15.17). We now show how a fundamental matrix $\underline{\mathcal{M}}(x,y)$ generally can be obtained from a transport matrix $\underline{M}(x,y)$. Rewrite (15.19) in the form

$$[\underbrace{\underline{H}(x,+),\underline{H}(y,-)}_{\substack{\text{transport}\\ \text{output}}},\ \underbrace{\underline{H}(y,+),\ \underline{H}(x,-)}_{\substack{\text{transport}\\ \text{input}}}]\begin{bmatrix}\underline{I}_{2m}\\ -\underline{M}(x,y)\end{bmatrix} = \underbrace{[\underline{H}_\eta(y,x),\ \underline{H}_\eta(x,y)]}_{\text{source-generated}} \tag{15.23}$$

On comparing this with (15.3), we see that the derivation of par. A can be carried out once again by generally following the steps from (15.4) to (15.17), but now starting with (15.23), and making appropriate notation changes. The details are left to the reader (cf. H.O., vol. IV, pp. 41,42). The end result is

$$\underline{M}(x,y) = \begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix} \tag{15.24}$$

$$= [\underline{C}_{-}\underline{M}(x,y) - \underline{C}_{+}][\underline{C}_{-} - \underline{C}_{+}\underline{M}(x,y)]^{-1} \tag{15.25}$$

$$= \left[\begin{array}{c:c} \underline{T}^{-1}(y,x) & \underline{T}^{-1}(y,x)\ \underline{R}(y,x) \\ \hdashline -\underline{R}(x,y)\ \underline{T}^{-1}(y,x) & \underline{T}(x,y) - \underline{R}(x,y)\ \underline{T}^{-1}(y,x)\ \underline{R}(y,x) \end{array}\right] \tag{15.26}$$

This solves in particular the problem of how to find $\underline{M}(a,x)$ from knowledge of

$$\underline{M}(a,x) \equiv \begin{bmatrix} \underline{t}(x,a) & \underline{r}(x,a) \\ \underline{r}(a,x) & \underline{t}(a,x) \end{bmatrix} \tag{15.27}$$

i.e., from knowledge of the reflectance and transmittance matrices of the air-water surface. Thus we have from (15.26) and (15.27):

$$\underline{M}(a,x) = \left[\begin{array}{c:c} \underline{t}^{-1}(x,a) & \underline{t}^{-1}(x,a)\ \underline{r}(x,a) \\ \hdashline -\underline{r}(a,x)\ \underline{t}^{-1}(x,a) & \underline{t}(a,x) - \underline{r}(a,x)\ \underline{t}^{-1}(x,a)\ \underline{r}(x,a) \end{array}\right] \tag{15.28}$$

The inverses occurring in (15.28) exist in the present case since $\underline{t}(a,x)$ and $\underline{t}(\underline{x},a)$ are diagonal matrices of nonzero elements. (There is no transpectral scatter inside the air-water surface.)

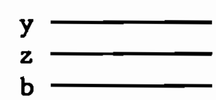
The transformation from $\underline{H}_{\eta}$ to $\underline{H}_{e}$ is also possible. The reverse of (15.18) is

$$[\underline{H}_e(x,y,+),\ \underline{H}_e(x,y,-)] = [\underline{H}_\eta(y,x),\ \underline{H}_\eta(x,y)]\ [\underline{C}_- - \underline{C}_+\underline{M}(x,y)]^{-1}$$

$$= [-\underline{H}_\eta(y,x)\ \underline{T}^{-1}(y,x),\ \underline{H}_\eta(x,y) - \underline{H}_\eta(y,x)\ \underline{T}^{-1}(y,x)\ \underline{R}(y,x)] \qquad (15.29)$$

C. *Fundamental to Transport Solution. Upward Case*

We consider next a submedium X(y,b) of X(a,b). The depth y is variable above the bottom boundary layer X(z,b). It is assumed that the incident irradiance vector $\underline{H}(b)$ along with $\underline{\mathcal{M}}(b,z)$ are known ($\underline{H}(b)$ follows, as shown, from the complete solution in section 14C. A method for finding $\underline{\mathcal{M}}(b,z)$ from the bottom boundary layer transfer matrices will be given below; cf. par. D). Then we can integrate (11.19) in the form



$$\frac{d}{dy}\ [\underline{H}(y,+),\ \underline{H}(y,-)] = [\underline{H}(y,+),\ \underline{H}(y,-)]\ \underline{\mathcal{K}}(y) + [-\underline{h}_e(y,+),\ \underline{h}_e(y,-)] \qquad (15.30)$$

from z to a higher variable depth y in X[x,z]. Since we know $\underline{H}(b)$ and $\underline{\mathcal{M}}(b,z)$, we then know $\underline{H}(z) = \underline{H}(b)\ \underline{\mathcal{M}}(b,z)$ in order to start the integration. By (14.18) we can write the general solution of (15.30) as

$$[\underline{H}(y,+),\ \underline{H}(y,-)] = [\underline{H}(z,+),\ \underline{H}(z,-)]\ \underline{\mathcal{M}}(z,y) + [\underline{H}_e(z,y,+),\ \underline{H}_e(z,y,-)] \qquad (15.31)$$

$$x \le y \le z$$

The reader will observe that the present discussion is building in a parallel way to that in par. A, above. Our next step is to write the present counterpart to (15.3):

$$[\underbrace{\underline{H}(y,+),\ \underline{H}(y,-)}_{\substack{\text{fundamental}\\ \text{output}}},\ \underbrace{\underline{H}(z,+),\ \underline{H}(z,-)}_{\substack{\text{fundamental}\\ \text{input}}}] \begin{bmatrix} \underline{I}_{2m} \\ \\ -\underline{M}(z,y) \end{bmatrix} = [\underbrace{\underline{H}_e(z,y,+),\ \underline{H}_e(z,y,-)}_{\text{source-generated}} \tag{15.32}$$

The present counterpart to (15.4) is

$$[\underbrace{\underline{H}(y,+),\underline{H}(y,-)}_{\substack{\text{fundamental}\\ \text{output}}},\underbrace{\underline{H}(z,+),\underline{H}(z,-)}_{\substack{\text{fundamental}\\ \text{input}}}] \underbrace{\begin{array}{c} \begin{array}{cc} \underline{C}_+ & \underline{C}_- \end{array} \\ \left[\begin{array}{cc|cc} \underline{I}_m & \underline{0}_m & \underline{0}_m & \underline{0}_m \\ \underline{0}_m & \underline{0}_m & \underline{0}_m & \underline{I}_m \\ \hline \underline{0}_m & \underline{0}_m & \underline{I}_m & \underline{0}_m \\ \underline{0}_m & \underline{I}_m & \underline{0}_m & \underline{0}_m \end{array}\right] \\ \begin{array}{cc} \underline{C}_- & \underline{C}_+ \end{array} \end{array}}_{\underline{Q}_+} = [\underbrace{\underline{H}(y,+),\underline{H}(z,-)}_{\substack{\text{transport}\\ \text{output}}},\underbrace{\underline{H}(z,+),\underline{H}(y,-)}_{\substack{\text{transport}\\ \text{input}}}] \tag{15.33}$$

Notice the change in the placement of $\underline{C}_+$ and $\underline{C}_-$ in the new version of $\underline{Q}$ relative to that in (15.4). This change is necessary to accommodate the new mapping between the present fundamental and transport input-output vectors. Observe that once again we have the important property

$$\begin{bmatrix} \underline{C}_+ & \underline{C}_- \\ \underline{C}_- & \underline{C}_+ \end{bmatrix}^2 = \underline{Q}_+^2 = \underline{I}_{4m} \tag{15.34}$$

Using $\underline{Q}_+^2$ in (15.32) and reducing the result in exactly the manner shown in par. A, we find the *upward global interaction principles*:

$$\boxed{\begin{array}{l} [\underline{H}(y,+),\ H(z,-)] = [\underline{H}(z,+),\ \underline{H}(y,-)]\ \underline{M}(y,z) + [\underline{H}_\eta(z,y),\ \underline{H}_\eta(y,z)] \\ y \le z \end{array}} \tag{15.35}$$

i.e.,

$$\boxed{\begin{aligned}
&\underline{H}(y,+) = \underline{H}(z,+)\,\underline{T}(z,y) + \underline{H}(y,-)\,\underline{R}(y,z) + \underline{H}_\eta(z,y)\\
&\underline{H}(z,-) = \underline{H}(z,+)\,\underline{R}(z,y) + \underline{H}(y,-)\,\underline{T}(y,z) + \underline{H}_\eta(y,z)\\
&y \le z
\end{aligned}}
\qquad
\begin{array}{ll}
a & \rule{2cm}{0.4pt}\\
x & \rule{2cm}{0.4pt}\\
y & \rule{2cm}{0.4pt}\\
z & \rule{2cm}{0.4pt}\\
b & \rule{2cm}{0.4pt}
\end{array}
\tag{15.36}$$

where

$$\underline{M}(y,z) \equiv \begin{bmatrix} \underline{T}(z,y) & \underline{R}(z,y) \\ \underline{R}(y,z) & \underline{T}(y,z) \end{bmatrix} \equiv [\underline{C}_+\,\underline{\mathcal{M}}(z,y) - \underline{C}_-][\underline{C}_+ - \underline{C}_-\underline{\mathcal{M}}(z,y)]^{-1} \tag{15.37}$$

$$= \left[\begin{array}{c:c} \underline{\mathcal{M}}_{++}(z,y) - \underline{\mathcal{M}}_{+-}(z,y)\,\underline{\mathcal{M}}_{--}^{-1}(z,y)\,\underline{\mathcal{M}}_{-+}(z,y) & -\underline{\mathcal{M}}_{+-}(z,y)\,\underline{\mathcal{M}}_{--}^{-1}(z,y) \\ \hdashline \underline{\mathcal{M}}_{--}^{-1}(z,y)\,\underline{\mathcal{M}}_{-+}(z,y) & \underline{\mathcal{M}}_{--}^{-1}(z,y) \end{array}\right] \tag{15.38}$$

The physical interpretation of (15.36) is similar to that of (15.20). The same clarity of meaning of each term holds here, too. Moreover, the source generated terms are related by writing

$$\text{'}[\underline{H}_\eta(z,y),\ \underline{H}_\eta(y,z)]\text{' for} \qquad\qquad (W\cdot m^{-2}\cdot nm^{-1})$$

$$[\underline{H}_e(z,y,+) - \underline{H}_e(z,y,-)\,\underline{\mathcal{M}}_{--}^{-1}(z,y)\,\underline{\mathcal{M}}_{-+}(z,y),\ -\underline{H}_e(z,y,-)\,\underline{\mathcal{M}}_{--}^{-1}(z,y)] \tag{15.39}$$

$$(= [\underline{H}_e(z,y,+),\ \underline{H}_e(z,y,-)][\underline{C}_+ - \underline{C}_-\ \underline{\mathcal{M}}(z,y)]^{-1})$$

Observe that $H_\eta(z,z) = \underline{0}$.

D. *Transport to Fundamental Solution. Upward Case*

The development here is parallel to that in par. B. We may therefore simply list the results. They are

$$\underline{M}(z,y) = \begin{bmatrix} \underline{M}_{++}(z,y) & \underline{M}_{+-}(z,y) \\ \underline{M}_{-+}(z,y) & \underline{M}_{--}(z,y) \end{bmatrix} \tag{15.40}$$

$$= [\underline{C}_+\underline{M}(z,y) - \underline{C}_-][\underline{C}_+ - \underline{C}_-\underline{M}(z,y)]^{-1} \tag{15.41}$$

$$= \left[\begin{array}{c:c} \underline{T}(z,y) - \underline{R}(z,y)\,\underline{T}^{-1}(y,z)\,\underline{R}(y,z) & -\underline{R}(z,y)\,\underline{T}^{-1}(y,z) \\ \hdashline \underline{T}^{-1}(y,z)\,\underline{R}(y,z) & \underline{T}^{-1}(y,z) \end{array}\right] \tag{15.42}$$

The source-generated terms for the upward case are related by

$$[\underline{H}_e(z,y,+),\ \underline{H}_e(z,y,-)] = [\underline{H}_\eta(z,y),\ \underline{H}_\eta(y,z)][\underline{C}_+ - \underline{C}_-\underline{M}(y,z)]^{-1}$$

$$= [\underline{H}_\eta(z,y) - \underline{H}_\eta(y,z)\,\underline{T}^{-1}(y,z)\,\underline{R}(y,z)\ ,\ -\underline{H}_\eta(y,z)\,T^{-1}(y,z)]$$

$$(15.43)$$

One can use (15.42), e.g., in the manner that (15.26) was used to find (15.28), i.e., to find the fundamental matrix of a surface in order, in this case, to propagate an irradiance pair ($\underline{H}(b,+)$, $\underline{H}(b,-)$) through X(b,x) from below. Thus, in (15.42), set $z \to b$, $y \to z$, and use to get ($\underline{H}(z,+)$, $\underline{H}(z,-)$ from ($\underline{H}(b,+)$,$\underline{H}(b,-)$).

E. *Symmetries of the $\underline{M}$-matrix*

The two-flow decomposition $(\underline{H}(y,+), \underline{H}(y,-))$ of the light field has several symmetries which, if specifically noted, can simplify various practical calculations of the field. One important source of simplifying formulas is the following symmetry property of $\underline{M}(x,z)$ on $X(x,z)$ (cf. (13.40)):

$$\underline{M}(x,z)\ \underline{M}(z,x) = \underline{I}_{2m} \tag{15.44a}$$

$$\underline{M}(z,x)\ \underline{M}(x,z) = \underline{I}_{2m} \tag{15.44b}$$

The first of these in block matrix form is

$$\begin{bmatrix} \underline{M}_{++}(x,z) & \underline{M}_{+-}(x,z) \\ \underline{M}_{-+}(x,z) & \underline{M}_{--}(x,z) \end{bmatrix} \begin{bmatrix} \underline{M}_{++}(z,x) & \underline{M}_{+-}(z,x) \\ \underline{M}_{-+}(z,x) & \underline{M}_{--}(z,x) \end{bmatrix} = \begin{bmatrix} \underline{I}_m & \underline{0}_m \\ \underline{0}_m & \underline{I}_m \end{bmatrix} \tag{15.45}$$

This yields four statements:

$$\underline{M}_{++}(x,z)\ \underline{M}_{++}(z,x) + \underline{M}_{+-}(x,z)\ \underline{M}_{-+}(z,x) = \underline{I}_m \tag{15.46a}$$

$$\underline{M}_{++}(x,z)\ \underline{M}_{+-}(z,x) + \underline{M}_{+-}(x,z)\ \underline{M}_{--}(z,x) = \underline{0}_m \tag{15.46b}$$

$$\underline{M}_{-+}(x,z)\ \underline{M}_{++}(z,x) + \underline{M}_{--}(x,z)\ \underline{M}_{-+}(z,x) = \underline{0}_m \tag{15.46c}$$

$$\underline{M}_{-+}(x,z)\ \underline{M}_{+-}(z,x) + \underline{M}_{--}(x,z)\ \underline{M}_{--}(z,x) = \underline{I}_m \tag{15.46d}$$

From (15.44b) we see that we can interchange 'x' and 'z' in (15.46) to obtain another valid set of statements.

F. *Symmetries of the $\underline{M}$-matrix*

Equation (15.17) for $\underline{M}(x,y)$ and equation (15.38) for $\underline{M}(y,z)$ give us useful formulas for the four transfer functions $\underline{R}_{\pm},\underline{T}_{\pm}$ of layers $X(x,y)$ and $X(y,z)$ in terms of the block submatrices of the fundamental matrices $\underline{\mathit{M}}(x,y)$ and $\underline{\mathit{M}}(z,y)$ respectively. If we set $y \to z$ in $\underline{\mathit{M}}(x,y)$ in (15.17) and $y \to x$ in $\underline{\mathit{M}}(y,z)$ in (15.38), we obtain two apparently distinct versions of $\underline{M}(x,z)$ from (15.17) and (15.38). For example, $\underline{T}(z,x)$ from (15.17) has a simple representation while $\underline{T}(z,x)$ from (15.38) appears complex. Our purpose here is to show that these apparently distinct representations are identical. The net result of the demonstration is the following compact representations of the four transfer functions of a layer $X(x,z)$:

$$\underline{T}(z,x) = \underline{\mathit{M}}_{++}^{-1}(x,z) \tag{15.47a}$$

$$\underline{T}(x,z) = \underline{\mathit{M}}_{--}^{-1}(z,x) \tag{15.47b}$$

$$\begin{aligned}\underline{R}(z,x) &= \underline{\mathit{M}}_{++}^{-1}(x,z)\ \underline{\mathit{M}}_{+-}(x,z)\\ &= -\underline{\mathit{M}}_{+-}(z,x)\ \underline{\mathit{M}}_{--}^{-1}(z,x)\end{aligned} \tag{15.47c}$$

$$\begin{aligned}\underline{R}(x,z) &= \underline{\mathit{M}}_{--}^{-1}(z,x)\ \underline{\mathit{M}}_{-+}(z,x)\\ &= -\underline{\mathit{M}}_{-+}(x,z)\ \underline{\mathit{M}}_{++}^{-1}(x,z)\end{aligned} \tag{15.47d}$$

Observe how the two $\underline{T}$ matrices can be obtained by interchanging 'x' and 'z' on the left and on the right, along with an interchange of '+' and '-'. A similar observation holds for the reflectance matrices.

We shall now demonstrate two of the equalities in (15.47). Equation (15.47c) comes first of all from a formal rearrangement of (15.46b):

$$\underline{M}_{++}^{-1}(x,z)\ \underline{M}_{+-}(x,z) = -\underline{M}_{+-}(z,x)\ \underline{M}_{--}^{-1}(z,x) \tag{15.48}$$

and then comparison of $\underline{R}(y,x)$ and $\underline{R}(z,y)$ in (15.17) and (15.38) in which have been made the substitutions $y \to z$ and $y \to x$, respectively. The additional representation of $R(z,x)$ is added for later reference.

To establish (15.47a), we rewrite (15.48) as

$$M_{++}^{-1}(x,z) = -\underline{M}_{+-}(z,x)\ \underline{M}_{--}^{-1}(z,x)\ \underline{M}_{+-}^{-1}(x,z) \tag{15.49}$$

Moreover, from (15.46a),

$$M_{++}^{-1}(x,z) = \underline{M}_{++}(z,x) + M_{++}^{-1}(x,z)\ \underline{M}_{+-}(x,z)\ \underline{M}_{-+}(z,x) \tag{15.50}$$

Next, use the representation of $M_{++}^{-1}(x,z)$ in (15.49) to replace $\underline{M}_{++}^{-1}(x,z)$ on the right side of (15.50). The result is

$$\underline{M}_{++}^{-1}(x,z) = \underline{M}_{++}(z,x) - \underline{M}_{+-}(z,x)\ \underline{M}_{--}^{-1}(z,x)\ M_{-+}(z,x) \tag{15.51}$$

Then use (15.51) to compare $\underline{T}(y,x)$ and $\underline{T}(z,y)$ in (15.17) and (15.38) in which have been made the substitutions $y \to z$ and $y \to x$, respectively. This establishes (15.47a).

The remaining two representations (15.47b) and (15.47d) can be obtained by the 'x' and 'z' interchange rule, in corresponding representations (e.g. the first representations in (15.47c), (15.47d)), or by using the remaining pair (15.46c), (15.46d) of symmetries analogously to the way just shown for the pair (15.46a), (15.46b).

We finally observe that the following simple representations of the source generated irradiances $\underline{H}_\eta(x,z)$ and $\underline{H}_\eta(z,x)$ are valid:

$$\boxed{\begin{aligned} \underline{H}_\eta(z,x) &= \underline{H}_e(x,z,+)\ \underline{M}_{++}^{-1}(x,z) \\ \underline{H}_\eta(x,z) &= -\underline{H}_e(z,x,-)\ \underline{M}_{--}^{-1}(z,x) \end{aligned}} \quad \begin{aligned} &(15.52a) \\ &(15.52b) \end{aligned}$$

These are checked by comparison of (15.18) and (15.39) on making the substitutions $y \to z$ and $y \to x$, respectively. Recall that $\underline{H}_e(u,v)$ was defined in (14.16) in terms of the source density $\underline{h}_e(s)$, and has the basic symmetry (14.40).

G. *Symmetries of the* $\underline{E}$*-matrix*

The $\underline{E}$ matrix introduced in (13.8), defined in (13.14), and partitioned in (13.19), shares the following symmetry with the inverse $\underline{F}$:

$$\underline{E}\ \underline{F} = \underline{F}\ \underline{E} = \underline{I}_{2m} \tag{15.53}$$

In particular from

$$\begin{bmatrix} \underline{F}_{++} & \underline{F}_{+-} \\ \underline{F}_{-+} & \underline{F}_{--} \end{bmatrix} \begin{bmatrix} \underline{E}_{++} & \underline{E}_{+-} \\ \underline{E}_{-+} & \underline{E}_{--} \end{bmatrix} = \begin{bmatrix} \underline{I}_m & \underline{0}_m \\ \underline{0}_m & \underline{I}_m \end{bmatrix} \tag{15.54}$$

we have the four statements

$$\underline{F}_{++}\,\underline{E}_{++} + \underline{F}_{+-}\,\underline{E}_{-+} = \underline{I}_m \tag{15.55a}$$

$$\underline{F}_{++}\,\underline{E}_{+-} + \underline{F}_{+-}\,\underline{E}_{--} = \underline{0}_m \tag{15.55b}$$

$$\underline{F}_{-+}\,\underline{E}_{++} + \underline{F}_{--}\,\underline{E}_{-+} = \underline{0}_m \tag{15.55c}$$

$$\underline{F}_{-+}\,\underline{E}_{+-} + \underline{F}_{--}\,\underline{E}_{--} = \underline{I}_m \tag{15.55d}$$

From (15.53) we see that we can interchange '$\underline{F}$' and '$\underline{E}$' in (15.55) to obtain another set of statements.

H. *Eigenmatrix Representations of the Transfer Matrices $\underline{R}$ and $\underline{T}$*

Let X(x,z) be a layer of lake or sea in which the optical properties $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$ are independent of depth. By combining (15.47) and (13.42) (in which we have set $y \to z$) we find the following representations of the four transfer matrices for X(x,z):

$$\underline{T}(z,x) = [\underline{E}_{++} e^{\underline{k}_+(z-x)}\, \underline{F}_{++} + \underline{E}_{+-} e^{\underline{k}_-(z-x)}\, \underline{F}_{-+}]^{-1} \tag{15.56a}$$

$$\underline{T}(x,z) = [\underline{E}_{-+} e^{\underline{k}_+(x-z)}\, \underline{F}_{+-} + \underline{E}_{--} e^{\underline{k}_-(x-z)}\, \underline{F}_{--}]^{-1} \tag{15.56b}$$

$$\underline{R}(z,x) = [\underline{E}_{++} e^{\underline{k}_+(z-x)}\, \underline{F}_{++} + \underline{E}_{+-} e^{\underline{k}_-(z-x)}\, \underline{F}_{-+}]^{-1}\, [\underline{E}_{++} e^{\underline{k}_+(z-x)}\, \underline{F}_{+-} + \underline{E}_{+-} e^{\underline{k}_-(z-x)}\, \underline{F}_{--}] \tag{15.56c}$$

$$\underline{R}(x,z) = [\underline{E}_{-+} e^{\underline{k}_+(x-z)}\, \underline{F}_{+-} + \underline{E}_{--} e^{\underline{k}_-(x-z)}\, \underline{F}_{--}]^{-1}\, [\underline{E}_{-+} e^{\underline{k}_+(x-z)}\, \underline{F}_{++} + \underline{E}_{--} e^{\underline{k}_-(x-z)}\, \underline{F}_{-+}] \tag{15.56d}$$

It is useful to have the expression for $\underline{R}(x,\infty)$, the reflectance for an infinitely deep homogeneous water layer whose upper surface is at level x. This is obtained from (15.56d) on letting z increase without bound and recalling the ordering (13.15) of the eigenvalues of $\underline{K}$. The result is, for all real media (i.e., with $k_-(1) < 0 < k_+(1)$ in (13.15)),

$$\lim_{z\to\infty} \underline{R}(x,z) \equiv \underline{R}(x,\infty) = \underline{\mathcal{F}}_{--}^{-1}\, \underline{\mathcal{F}}_{-+} = -\underline{\mathcal{E}}_{-+}\underline{\mathcal{E}}_{++}^{-1} \tag{15.57}$$

which is independent of x.

The second version involving the $\underline{\mathcal{E}}$ matrix comes from (15.55c). Observe in a similar way, now from (15.56c), that

$$\lim_{z\to\infty} \underline{R}(z,x) \equiv \underline{R}(\infty,x) = \underline{\mathcal{F}}_{++}^{-1}\, \underline{\mathcal{F}}_{+-} = -\underline{\mathcal{E}}_{+-}\underline{\mathcal{E}}_{--}^{-1} \tag{15.58}$$

Also, in all real homogeneous media we have from (15.56a,b) that

$$\lim_{z\to\infty} \underline{T}(x,z) = \lim_{z\to\infty} \underline{T}(z,x) = \underline{0}_m \tag{15.59}$$

I. *Small-Depth Representations of $\underline{R}$ and $\underline{T}$.*

The forms of (15.56) for small depth differences z-x can be determined if we first note the connections between the elements of $\underline{\mathcal{K}}$ and $\underline{\mathcal{E}}$ in (13.14). Write (13.14) in the form

$$\underline{\mathcal{K}} = \underline{\mathcal{E}}\ \underline{k}\ \underline{\mathcal{F}} \tag{15.60}$$

which, on expanding the right side, yields

$$\begin{bmatrix} -\underline{\tau}(+) & \underline{\rho}(+) \\ -\underline{\rho}(-) & \underline{\tau}(-) \end{bmatrix} = \left[\begin{array}{c:c} \underline{\mathcal{E}}_{++}\,\underline{k}_{+}\,\underline{\mathcal{F}}_{++} + \underline{\mathcal{E}}_{+-}\,\underline{k}_{-}\,\underline{\mathcal{F}}_{-+} & \underline{\mathcal{E}}_{++}\,\underline{k}_{+}\,\underline{\mathcal{F}}_{+-} + \underline{\mathcal{E}}_{+-}\,\underline{k}_{-}\,\underline{\mathcal{F}}_{--} \\ \hdashline \underline{\mathcal{E}}_{-+}\,\underline{k}_{+}\,\underline{\mathcal{F}}_{++} + \underline{\mathcal{E}}_{--}\,\underline{k}_{-}\,\underline{\mathcal{F}}_{-+} & \underline{\mathcal{E}}_{-+}\,\underline{k}_{+}\,\underline{\mathcal{F}}_{+-} + \underline{\mathcal{E}}_{--}\,\underline{k}_{-}\,\underline{\mathcal{F}}_{--} \end{array}\right] \tag{15.61}$$

In this way we can represent the local optical properties $(\underline{\rho},\underline{\tau})$ in terms of the eigenstructures $(\underline{E},\underline{k})$ of the matrix $\underline{K}$.

Now, in (15.56) for small z-x, we use the first order approximation

$$\exp[\underline{k}_{\pm}(z-x)] \approx \underline{I}_m + \underline{k}_{\pm}(z-x) \tag{15.62}$$

Then (15.56), on making use of (15.55), reduces with the help of (15.61), to the following first order approximations in (z-x):

$$\underline{T}(z,x) = \underline{I}_m + \underline{\tau}(+)(z-x) \tag{15.63a}$$

$$\underline{T}(x,z) + \underline{I}_m + \underline{\tau}(-)(z-x) \tag{15.63b}$$

$$\underline{R}(z,x) = \underline{\rho}(+)(z-x) \tag{15.63c}$$

$$\underline{R}(x,z) = \underline{\rho}(-)(z-x) \tag{15.63d}$$

This shows the connection between the local $(\underline{\rho},\underline{\tau})$ and global $(\underline{R},\underline{T})$ transfer functions for thin layers of hydrosol. Although these connections have been made for homogeneous media, they are generally valid for depths y in X(x,z), z-x small, in stratified media. The same connection can be viewed in the fundamental matrix setting by writing (13.37) as

$$\begin{aligned}
\underline{H}(z) &= \underline{H}(x)\ \underline{M}(x,z) \\
&= \underline{H}(x)\ \underline{E}\ \exp[\underline{k}(z-x)]\underline{F} \\
&\approx \underline{H}(x)\ \underline{E}[\underline{I}_{2m} + \underline{k}(z-x)]\underline{F} \\
&= \underline{H}(x)[\underline{E}\ \underline{F} + \underline{E}\ \underline{k}\ \underline{F}(z-x)] \\
&= \underline{H}(x)\ [\underline{I}_{2m} + \underline{K}(z-x)]
\end{aligned} \tag{15.64}$$

The final reduction was made using (13.14).

Hence to the first order in (z-x),

$$\underline{M}(x,z) = \begin{bmatrix} \underline{M}_{++}(x,z) & \underline{M}_{+-}(x,z) \\ \underline{M}_{-+}(x,z) & \underline{M}_{--}(x,z) \end{bmatrix} = \left[\begin{array}{c:c} \underline{I}_m - \underline{\tau}(+)(z-x) & \underline{\rho}(+)(z-x) \\ \hdashline -\underline{\rho}(-)(z-x) & \underline{I}_m + \underline{\tau}(-)(z-x) \end{array}\right] \tag{15.65}$$

This is consistent, to within first order in (z-x), with (15.47) and (15.63).

J. *Large-Depth Representations of* $\underline{R}$

We close this discussion of the eigenmatrix theory of the global transfer functions by remarking that the theory of directly observable relations in sec. 12 (which was developed for arbitrarily stratified media) can be given a parallel development in homogeneous media. Because the eigenstructures $\underline{E}$ and $\underline{k}$ and $\underline{K}$ are available in homogeneous media, the various generalizations from the monochromatic to the heterochromatic level discussed in section 12, can be cast into particularly detailed form using eigenstructure concepts. We shall not enter here into a detailed development of the reductions of the results in section 12 to homogeneous media; but an illustration of the preceding remark will be made.

Reconsider the $\underline{R}$-$\underline{K}$ connection (12.26). It was observed below (12.26) that such a connection is particularly useful in deep homogeneous media. Thus we may ask: what is the form of (12.26) in such media when the eigenmatrix formulation is available? Now, from (15.61) we may write

$$\underline{\rho}(+) = \underline{E}_{++}\, \underline{k}_{+}\, \underline{F}_{+-} + \underline{E}_{+-}\, \underline{k}_{-}\, \underline{F}_{--} \tag{15.66}$$

Keeping (15.57) in mind we multiply (15.66) on the left by $-\underline{E}_{-+}\,\underline{E}_{++}^{-1}$ to find

$$\underline{R}(x,\infty)\ \underline{\rho}(+) = -\underline{E}_{-+}\ \underline{k}_{+}\ \underline{F}_{+-} - \underline{E}_{-+}\ \underline{E}_{++}^{-1}\ \underline{E}_{+-}\ \underline{k}_{-}\ \underline{F}_{--} \qquad (15.67)$$

Adding and subtracting $\underline{E}_{--}\ \underline{k}_{-}\ \underline{F}_{--}$ on the right side of (15.67), and recalling (15.61), we have

$$\underline{R}(x,\infty)\ \underline{\rho}(+) = -\underline{\tau}(-) + [\underline{E}_{--} - \underline{E}_{-+}\ \underline{E}_{++}^{-1}\ \underline{E}_{+-}]\ \underline{k}_{-}\ \underline{F}_{--} \qquad (15.68)$$

Applying (15.55) to the matrix combination in the square brackets of (15.68), we recognize that combination as being $\underline{F}_{--}^{-1}$. Hence we arrive at the result

$$\boxed{\underline{R}(x,\infty) = [\underline{F}_{--}^{-1}\ \underline{k}_{-}\ \underline{F}_{--} - \underline{\tau}(-)]\ \underline{\rho}^{-1}(+)} \qquad (15.69)$$

which is the eigenmatrix version of the second form in (12.26). Here $\underline{F}_{--}^{-1}\ \underline{k}_{-}\ \underline{F}_{--}$ plays the role of $-\underline{K}(y,-)$. The result (15.69) is useful in that it represents the global property $R(x,\infty)$ by the local concepts $\underline{\rho}(+)$, $\underline{\tau}(-)$ and $\underline{k}$. Using (15.69) in (12.19) when we have an infinitely deep homogeneous hydrosol, we find that

$$\begin{aligned}\underline{K}(y,-) &= -[\underline{\tau}(-) + \underline{R}(y,\infty)\underline{\rho}(+)]\\ &= -[\underline{\tau}(-) + \underline{F}_{--}^{-1}\ \underline{k}_{-}\ \underline{F}_{--} - \tau(-)]\end{aligned}$$

Hence, on dropping y from $\underline{K}(y,-)$, we have

$$\boxed{\underline{K}(-) = -\underline{\mathcal{F}}_{--}^{-1}\, \underline{k}_{-}\, \underline{\mathcal{F}}_{--}} \tag{15.70}$$

For reference we add the upward companions to (15.69), (15.70):

$$\underline{R}(\infty,x) = -[\underline{\mathcal{F}}_{++}^{-1}\, \underline{k}_{+}\, \underline{\mathcal{F}}_{++} + \underline{\tau}(+)]\, \underline{\rho}^{-1}(-) \tag{15.71}$$

$$\underline{K}(+) = -\underline{\mathcal{F}}_{++}^{-1}\, \underline{k}_{+}\, \underline{\mathcal{F}}_{++} \tag{15.72}$$

Here $\underline{R}(\infty,x)$ is interpretable as the observed upward reflectance for an infinitely deep homogeneous medium when x is very deep. Then compare the inverse of (15.71):

$$\underline{R}^{-1}(\infty,x) = \underline{\rho}(-)[-\underline{\mathcal{F}}_{++}\, \underline{k}_{+}\, \underline{\mathcal{F}}_{++} - \underline{\tau}(+)]^{-1} \tag{15.73}$$

with the first form of (12.26).

16. DIFFERENTIAL EQUATIONS FOR $\underline{R}$, $\underline{T}$, $\underline{H}_\eta$ (RICCATI SWEEPS) AND $\underline{A}$.

We will next show how the light field in a natural hydrosol may be determined using the global interaction principles of section 15.

The discussion will take place in two main stages. First in par. A, we consider the downward sweep procedure for generating the $\underline{R}$, $\underline{T}$, and $\underline{H}_\eta$ functions of a lake or sea. Then in par. B the upward sweep procedure will be developed. By means of these two approaches to $\underline{R}$, $\underline{T}$, and $\underline{H}_\eta$ the light field in X(a,b) may generally be determined.

A. *Downward (Riccati) Sweep*

The setting for the present discussion is the optical medium X(a,b) shown in the lower left sketch of Fig. 7. The light field existing in X(a,b) is initiated and sustained by arbitrary but fixed incident boundary irradiances $\underline{H}(a,-)$, $\underline{H}(b,+)$ and arbitrary but fixed internal emission source densities $\underline{h}_e(y,\pm)$, $x \le y \le z$. We imagine an irradiance probe in the internal part X(x,z) of X(a,b) measuring $\underline{H}(y,\pm)$ at all levels y, $x \le y \le z$ without disturbing the light field. The probe moves downward, say, through X(x,z), starting at level x, just below the upper boundary. As we continuously increase the depth y, the monitored light field values $\underline{H}(y,\pm)$, $\underline{H}_\eta(y,x)$, $\underline{H}_\eta(x,y)$ and the four transfer matrices $\underline{R}(y,x)$, $\underline{T}(x,y)$, $\underline{R}(x,y)$, $\underline{T}(y,x)$ change value so as always to have (15.20) hold. Therefore the y-derivatives of each equation in (15.20) hold:

$$\underline{0} = \underline{H}'(y,+)\, T(y,x) + \underline{H}(y,+)\, \underline{T}'(y,x) + H(x,-)\, \underline{R}'(x,y) + \underline{H}_\eta'(y,x) \qquad (16.1)$$

$$\underline{H}'(y,-) = \underline{H}'(y,+)\, \underline{R}(y,x) + \underline{H}(y,+)\, \underline{R}'(y,x) + \underline{H}(x,-)\, \underline{T}'(x,y) + \underline{H}_\eta'(x,y) \qquad (16.2)$$

where primes denote differentiation with respect to y. These equations may be simplified by using (11.19) to replace the derivatives of $\underline{H}(y,\pm)$. For example, (16.2) becomes

$$[\underline{H}(y,-)\ \underline{\tau}(y,-) + \underline{H}(y,+)\ \underline{\rho}(y,+) + \underline{h}_e(y,-)]$$

$$= -[\underline{H}(y,+)\ \underline{\tau}(y,+) + \underline{H}(y,-)\ \underline{\rho}(y,-) + \underline{h}_e(y,+)]\ \underline{R}(y,x)$$

$$+ \underline{H}(y,+)\ \underline{R}'(y,x) + \underline{H}(x,-)\ \underline{T}'(x,y) + \underline{H}'_\eta(x,y) \tag{16.3}$$

Next, in (16.3) we replace the response irradiance $\underline{H}(y,-)$ of $X(x,y)$ by means of (15.20), and then collect together the coefficients of the incident irradiances $\underline{H}(x,-)$, $\underline{H}(y,+)$ of $X(x,y)$, and also collect together the emission-source related terms. The result is that the transformed (16.2) may be regrouped into three main parts whose sum is the zero vector:

$$\underline{H}(y,+)\ [-\underline{R}'(y,x) + \underline{\rho}(y,+) + \underline{\tau}(y,+)\ \underline{R}(y,x) + \underline{R}(y,x)\ \underline{\tau}(y,-) + \underline{R}(y,x)\ \underline{\rho}(y,-)\ \underline{R}(y,x)]$$

$$+ \underline{H}(x,-)\ [-\underline{T}'(x,y) + \underline{T}(x,y)\ \underline{\tau}(y,-) + \underline{T}(x,y)\ \underline{\rho}(y,-)\ \underline{R}(y,x)]$$

$$+ [-\underline{H}'_\eta(x,y) + \underline{H}_\eta(x,y)\ \underline{\tau}(y,-) + \underline{H}_\eta(x,y)\ \underline{\rho}(y,-)\ \underline{R}(y,x) + \underline{h}_e(y,-) + \underline{h}_e(y,+)\ \underline{R}(y,x)] = \underline{0} \tag{16.4}$$

Returning now to (16.1) we replace the derivative $\underline{H}'(y,+)$ by means of (11.19) and the newly appearing response irradiance $\underline{H}(y,-)$ by means of (15.20). Then collecting together the coefficients of the incident irradiances $\underline{H}(x,-)$, $\underline{H}(y,+)$ of $X(x,y)$ and also collecting together the emission-source related terms, we find that the transformed (16.1) becomes

$$\underline{H}(y,+)\ [\underline{T}'(y,x) - \underline{\tau}(y,+)\ \underline{T}(y,x) - \underline{R}(y,x)\ \underline{\rho}(y,-)\ \underline{T}(y,x)]$$

$$+\ \underline{H}(x,-)\ [\underline{R}'(x,y) - \underline{T}(x,y)\ \underline{\rho}(y,-)\ \underline{T}(y,x)]$$

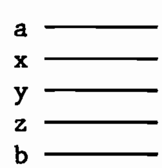
$$+\ [\underline{H}_\eta'(y,x) - \underline{h}_e(y,+)\ \underline{T}(y,x) - \underline{H}_\eta(x,y)\ \underline{\rho}(y,-)\ \underline{T}(y,x)] = \underline{0} \qquad (16.5)$$

We next note that the incident irradiances $H(x,-)$ and $H(y,+)$ on $X(x,y)$ may be arbitrarily varied in size with the two sets of equations (16.4), (16.5) remaining valid. Moreover, the s-dependence of the internal emission source densities $\underline{h}_e(s,\pm)$ throughout $X(x,y)$ may be varied independently of these incident irradiances. (Think of $X(x,y)$ as an isolated medium illuminated by three *independent* light sources: $\underline{H}(x,-)$, $\underline{H}(y,+)$ and $\underline{h}_e(s,\pm)$, $x \le s \le y$.) Hence the coefficients of $H(y,+)$, $\underline{H}(x,-)$ in (16.4) and in (16.5) must vanish along with the source-related term. We therefore arrive at the desired *downward (Riccati) sweep sextet* of differential equations:

$$\frac{\partial \underline{R}(y,x)}{\partial y} = \underline{R}(y,x)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,x)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\ \underline{R}(y,x)] \qquad (16.6)$$

$$\frac{\partial \underline{T}(x,y)}{\partial y} = \underline{T}(x,y)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,x)] \qquad (16.7)$$

$$\frac{\partial \underline{H}_\eta(x,y)}{\partial y} = \underline{H}_\eta(x,y)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,x)] + [\underline{h}_e(y,-) + \underline{h}_e(y,+)\ \underline{R}(y,x)] \qquad (16.8)$$



$$\frac{\partial \underline{T}(y,x)}{\partial y} = [\underline{\tau}(y,+) + \underline{R}(y,x)\ \underline{\rho}(y,-)]\ \underline{T}(y,x) \tag{16.9}$$

$$\frac{\partial \underline{R}(x,y)}{\partial y} = \underline{T}(x,y)\ \underline{\rho}(y,-)\ \underline{T}(y,x) \tag{16.10}$$

$$\frac{\partial \underline{H}_\eta(y,x)}{\partial y} = [\underline{h}_e(y,+) + \underline{H}_\eta(x,y)\ \rho(y,-)]\ \underline{T}(y,x) \tag{16.11}$$

Observe that the first trio of equations is autonomous (self-contained) while the second trio uses the $\underline{R}$ and $\underline{T}$ functions of the first trio in order to propagate along the sweep downward. We call the first three equations the *major trio* while the remaining ones make up the *minor trio*.

Observe also that of all six differential equations the one for $\underline{R}(y,x)$ is autonomous, and has the classical form of a Riccati differential equation, but now on the matrix level. In this sense $\underline{R}(y,x)$ is the most powerful transfer matrix of the quartet appearing in the downward global interaction equations (15.20). The entire sextet (16.6)-(16.11) can be integrated essentially simultaneously provided the integration of $\underline{R}(y,x)$'s equation takes place one integration step ahead of the other two in the major trio and that these two laggards of the major trio stay one step ahead of each member of the minor trio. In this sense $\underline{R}(y,x)$'s equation pulls all five other equations along with it. Hence the name 'Riccati Sweep' applied to the integration of the sextet (16.6)-(16.11) or its major trio (16.6)-(16.8).

The matter of the initial values of the $\underline{R}$, $\underline{T}$, $\underline{H}_\eta$ equations will be taken up next. Suppose first that we wish to integrate the sextet (16.6)-(16.11) downward through the *bare* slab X(x,z), i.e., X(x,z) consists only of water from which its optically active air-water surface and reflecting bottom have been (conceptually) peeled off. Then the initial values are

$$
\begin{aligned}
&\underline{R}(x,x) = \underline{0}_m && (m\times m)\\
&\underline{T}(x,x) = \underline{I}_m && (m\times m)\\
&\underline{H}_\eta(x,x) = \underline{0} && (1\times m)
\end{aligned}
\qquad (16.12)
$$

If subsequently we wish to find the transfer functions for $X(a,y) = X(a,x) \cup X(x,y)$, where $a \le x < y$, then we can use the union rule of section 18 to combine the given transfer functions of the boundary $X(a,x)$ and those of the bare slab $X(x,y)$ just found.

The transfer functions for $X(a,y) = X(a,x) \cup X(x,y)$, for the case of the air-water surface plus a variable water layer, can be found also by using the initial conditions

$$
\left.
\begin{aligned}
\underline{R}(x,a) &= \underline{r}(x,a)\\
\underline{R}(a,x) &= \underline{r}(a,x)\\
\underline{T}(a,x) &= \underline{t}(a,x)\\
\underline{T}(x,a) &= \underline{t}(x,a)\\
\underline{H}_\eta(a,x) &= \underline{0}\\
\underline{H}_\eta(x,a) &= \underline{0}
\end{aligned}
\right\}
\qquad (16.13)
$$

and then integrating (16.6)-(16.11) downward from level x to level y in $X(x,z)$. For example, in (16.6) we make the substitution $x \to a$, and the initial value for $R(y,a)$ is $\underline{r}(x,a)$. In (16.7), once again we set $x \to a$, and the initial value of $\underline{T}(a,y)$ is $\underline{t}(a,x)$; and so on for the remaining four equations. In the case of $X(a,x)$ being the air-water surface, $\underline{H}_\eta(a,x)$ and $\underline{H}_\eta(x,a)$ are zero vectors; in general when $X(a,x)$ contains the sources, these values must be specified to start the downward integrations of (16.8) and (16.11) at level x.

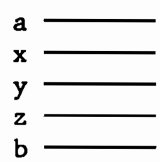
B. *Upward (Riccati) Sweep*

The setting for the present discussion is the optical medium X(a,b) shown in the lower right sketch of Fig. 7. As in par. A, the light field existing in X(a,b) is initiated and sustained by incident boundary irradiances $\underline{H}(a,-)$, $\underline{H}(b,+)$ along with the emission source densities $\underline{h}_e(y,\pm)$, $x \le y \le z$. The remainder of the present derivation proceeds as in par. A. Here is its outline: We imagine an irradiance probe in X(x,z) measuring $\underline{H}(y,\pm)$ as it moves upward from z to y in X(x,z), $x \le y \le z$. Equations (15.36) describe the light field and $\underline{R}$, $\underline{T}$ values at each y. Hence we can differentiate with respect to y each side of the equations in (15.36), and replace occurrences of $dH(y,\pm)/dy$ by means of (11.19), and go on to replace occurrences of the response irradiance $\underline{H}(y,+)$ by means of (15.36), and then group each resultant transformed equation of (15.36) into three parts, namely the coefficients of the incident irradiances $\underline{H}(y,-)$, $\underline{H}(z,+)$ and the source-related terms to find two sets of linear combinations, like (16.4) and (16.5), set to the zero vector. By an analogous argument to that below (16.5) we arrive at the *upward (Riccati) sweep sextet* of differential equations:

$$-\frac{\partial \underline{R}(y,z)}{\partial y} = \underline{R}(y,z)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\underline{R}(y,z)] + [\underline{\rho}(y,-) + \underline{\tau}(y,-)\underline{R}(y,z)] \quad (16.14)$$

$$-\frac{\partial \underline{T}(z,y)}{\partial y} = \underline{T}(z,y)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\underline{R}(y,z)] \quad (16.15)$$

$$-\frac{\partial \underline{H}_\eta(z,y)}{\partial y} = \underline{H}_\eta(z,y)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\underline{R}(y,z)] + [\underline{h}_e(y,+) + \underline{h}_e(y,-)\underline{R}(y,z)] \quad (16.16)$$



$$-\frac{\partial \underline{T}(y,z)}{\partial y} = [\underline{\tau}(y,-) + \underline{R}(y,z)\underline{\rho}(y,+)]\ \underline{T}(y,z) \quad (16.17)$$

$$-\frac{\partial \underline{R}(z,y)}{\partial y} = \underline{T}(z,y)\underline{\rho}(y,+)\underline{T}(y,z) \quad (16.18)$$

$$-\frac{\partial \underline{H}_\eta(y,z)}{\partial y} = [\underline{h}_e(y,-) + \underline{H}_\eta(z,y)\underline{\rho}(y,+)]\ \underline{T}(y,z) \quad (16.19)$$

The first trio is autonomous relative to the second trio and is the *major trio* of the upward sweep sextet of differential equations; the second set of equations is the *minor trio*. The equation for $\underline{R}(y,z)$ is the autonomous relative to the remaining five equations and is precisely the equation governing $\underline{R}(y,-)$ in (12.11). Therefore $\underline{R}(y,z)$ and $\underline{R}(y,-)$ will be the same function of y in X(x,z) if they have the same initial value, say $\underline{r}(z,b)$, the reflectance of the lower boundary X(z,b) of the medium
X(a,b) = X(a,x) U X(x,z) U X(z,b).

The general case of integrating the sextet (16.14)-(16.19) upward from level z to variable levels y in the composite medium X(x,b) = X(x,z) U X(z,b) proceeds as in the downward case in par. A, above. Now one generally starts the upward sweep by setting $z \to b$ and using the initial values

$$\left.\begin{aligned}
\underline{R}(z,b) &= \underline{r}(z,b)\\
\underline{R}(b,z) &= \underline{r}(b,z)\\
\underline{T}(b,z) &= \underline{t}(b,z)\\
\underline{T}(z,b) &= \underline{t}(z,b)\\
\underline{H}_\eta(b,z) &= \underline{0}\\
\underline{H}_\eta(z,b) &= \underline{0}
\end{aligned}\right\} \tag{16.20}$$

If the medium X(a,b) is a laboratory hydrosol in a tank with a translucent glass plate for a bottom X(z,b) then the full quartet $\underline{r}_\pm,\underline{t}_\pm$ in (16.20) may be needed and with $\underline{H}_\eta(b,z) = \underline{H}_\eta(z,b) = \underline{0}$ over X(z,b). If we have a natural, silty lake or ocean bottom at finite optical depth, then only $\underline{r}_- = \underline{r}(z,b)$ will be needed while the two initial values of the source irradiances $\underline{H}_\eta(b,z)$ and $\underline{H}_\eta(z,b)$ will be zero since no light is generated in the silty opaque layer. In this case we can also set $\underline{t}(z,b) = \underline{t}(b,z) = \underline{0}_m$. The matrix $\underline{r}(b,z)$, while definable, will not be needed in this case since upward incident flux at level b will not generally get through the opaque boundary to generate its share of the light field in X(x,z). If X(z,b) is simply another fluorescing layer of X(a,b) below X(x,z), then its four $\underline{R},\underline{T}$ matrices must be given, as in (16.20), along with the appropriate (nonzero) values of $\underline{H}_\eta(b,z)$ and $\underline{H}_\eta(z,b)$, before the upward sweep begins at level z.

C. *The Differential Equations for Global Absorption in Source-Free non-Fluorescing Media*

The set of global reflectance and transmittance functions of arbitrary source-free non-fluorescing slabs X[x,y] and X[y,z] growing either downward or upward at level y is completed by including the diagonal matrices $\underline{A}(x,y)$ and $\underline{A}(y,x)$ which are needed for a full statement of conservation of radiant energy absorbed, reflected and transmitted by these media. Thus we define A(x,y) and A(y,x) implicitly in terms of the $\underline{R}$ and $\underline{T}$ matrices by writing

$$\underline{A}(y,x) + \underline{R}(y,x) + \underline{T}(y,x) \equiv \underline{I} \tag{16.21}$$

and

$$\underline{A}(y,z) + \underline{R}(y,z) + \underline{T}(y,z) \equiv \underline{I} \tag{16.22}$$

where $\underline{I}$ is the m×m identity matrix. It is to be emphasized that $\underline{A}$, $\underline{R}$, and $\underline{T}$ are now diagonal matrices, by hypothesis. Each diagonal element of $\underline{A}(y,x)$ is the amount of radiant flux absorbed by X[x,y] for incident flux at level $y > x$.

The differential equation of $\underline{A}(y,x)$ for the downward sweep through X[x,y] at level y is obtained from (16.21) by differentiating each side of (16.21) with respect to y and then reducing and rearranging the $\underline{R}$ and $\underline{T}$ terms by means of their differential equations. The result, along with the associated equations for $\underline{R}$ and $\underline{T}$, is

$$\text{downward sweep}\left\{\begin{aligned}
&\frac{\partial \underline{A}(y,x)}{\partial y} = [\underline{a}(y,+) + \underline{R}(y,x)\,\underline{a}(y,-)] + [\underline{\tau}(y,+) + \underline{R}(y,x)\,\underline{\rho}(y,-)]\,\underline{A}(y,x) && (16.23)\\
&\frac{\partial \underline{R}(y,x)}{\partial y} = \underline{R}(y,x)\,[\underline{\tau}(y,-) + \underline{\rho}(y,-)\,\underline{R}(y,x)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\,\underline{R}(y,x)] && (16.6)\\
&\frac{\partial \underline{T}(y,x)}{\partial y} = [\underline{\tau}(y,+) + \underline{R}(y,x)\,\underline{\rho}(y,-)]\,\underline{T}(y,x) && (16.9)\\
&\underline{A}(x,x) = \underline{0} && (16.24)
\end{aligned}\right.$$

These equations each split into scalar equations, one for each diagonal element of the indicated matrices.

The interpretation of the terms on the right in (16.23) is interesting. The coefficient of $\underline{A}(y,x)$ is generally a negative term in absorbing media so that $\underline{A}(y,x)$ decays as y descends, with x fixed. What $\underline{A}(y,x)$ loses this way is similar to $\underline{T}(y,x)$'s type of loss, as may be seen by inspecting (16.9). The first term on the right in (16.23) is even more interesting. It represents absorbed flux gained by X[x,y] at level y when X[x,y] engulfs the next incremental layer as y descends. Hence such flux is added to the accumulated total $\underline{A}(y,x)$ of absorbed flux within X[x,y], and thereby represents a growth term for A(y,x).

The upward sweep generating A(y,z) is obtained from (16.22) in a similar way. The result, along with its two companions, is

upward sweep

$$-\frac{\partial \underline{A}(y,z)}{dy} = [\underline{a}(y,-) + \underline{R}(y,z)\ \underline{a}(y,+)] + [\underline{\tau}(y,-) + \underline{R}(y,z)\ \underline{\rho}(y,+)]\ \underline{A}(y,z) \qquad (16.25)$$

$$-\frac{\partial \underline{R}(y,z)}{dy} = \underline{R}(y,z)\ [\underline{\tau}(y,+) + \underline{\rho}(y,+)\ \underline{R}(y,z)] + [\underline{\rho}(y,-) + \underline{\tau}(y,-)\ \underline{R}(y,z)] \qquad (16.14)$$

$$-\frac{\partial \underline{T}(y,z)}{dy} = [\underline{\tau}(y,-) + \underline{R}(y,z)\ \underline{\rho}(y,+)]\ \underline{T}(y,z) \qquad (16.17)$$

$$\underline{A}(z,z) = \underline{0} \qquad (16.26)$$

When generating A(y,z) one needs only (16.14), and so (16.25) belongs to the sextet (16.11)-(16.19); while the differential equation for $\underline{A}(y,x)$ belongs to the sextet (16.6)-(16.11).

The preceding equations can form the basis for a full energy conservation statement in fluorescing media with true sources. We shall not need to go beyond this point in the present study. The preceding equations, however, as they stand, are of interest in the classic two-flow theory of non-fluorescing, source-free media, where they are immediately applicable in scalar form (i.e., single wavelength form).

17. IMBED RULE

The imbed rule, to be developed below, leads to the determination of the upward and downward irradiances $\underline{H}(y,\pm)$ *at any interior level* y of a water layer X(x,z), $x \le y \le z$ (cf. Fig. 7) when we know the incident irradiances $\underline{H}(x,-)$ and $\underline{H}(z,+)$ on the layer and the distribution of the emission sources in the layer. More generally, suppose we have a medium X(a,b) = X(a,x) U X(x,z) U X(z,b) where X(a,x) and X(a,b) are *boundary media* and X(x,z) is the *interior medium* of X(a,b). In the context of hydrologic optics, X(a,x) and X(z,b), e.g., are the air-water surface and lake or sea bottom, respectively. The imbed rule in such a setting finds $\underline{H}(y,\pm)$ at any level y of the interior medium X(x,z). For didactic reasons we will consider separately the imbed rule applied first to the interior medium X(x,z) and then to the composite medium X(a,b). Throughout this section (and section 18, below) we assume that the optical properties $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$ and $\underline{h}_e(y,\pm)$ are continuous functions of y over X(x,z). We call such an X(x,z) a *bare slab*, since it has no optically active zero-thickness planes within it or bounding it. We develop two approaches to the imbed rule. The first approach is algebraic and is based on the global interaction principles; the second is analytic and is based on the local interaction principles. We shall consider these approaches in turn, as we ascend through a series of ever more general formulations.

A. *Algebraic Imbedding in* X(x,z) = X(x,y) U (y,z)

We start with the global principles representing $\underline{H}(y,\pm)$. At the interface level y of two contiguous bare slabs X(x,y) and X(y,z) we have (via (15.36)) $$\underline{H}(y,+) = \underline{H}(z,+)\ \underline{T}(z,y) + \underline{H}(y,-)\ R(y,z) + \underline{H}_\eta(z,y) \tag{17.1}$$

(via (15.20)) $$\underline{H}(y,-) = \underline{H}(y,+)\ \underline{R}(y,x) + \underline{H}(x,-)\ \underline{T}(x,y) + \underline{H}_\eta(x,y) \tag{17.2}$$

Substituting $\underline{H}(y,-)$ as given by (17.2), into (17.1) and solving for $\underline{H}(y,+)$, and similarly solving for $\underline{H}(y,-)$ by the reverse substitution of (17.1) into (17.2), we find the desired *imbed rule* (or *invariant imbedding relation*):

$$\underline{H}(y,+) = \underline{H}(z,+)\ \underline{\mathcal{T}}(z,y,x) + \underline{H}(x,-)\ \underline{\mathcal{R}}(x,y,z) + \underline{\mathcal{H}}_\eta(z,y,x) \tag{17.3}$$

$$\underline{H}(y,-) = \underline{H}(z,+)\ \underline{\mathcal{R}}(z,y,x) + \underline{H}(x,-)\ \underline{\mathcal{T}}(x,y,z) + \underline{\mathcal{H}}_\eta(x,y,z) \tag{17.4}$$

$$x \le y \le z$$

We define the *complete upward and downward transmittance and reflectance* operators as

$$\underline{\mathcal{T}}(z,y,x) \equiv \underline{T}(z,y)[\underline{I}_m - \underline{R}(y,x)\ \underline{R}(y,z)]^{-1} \tag{17.5}$$

$$\underline{\mathcal{R}}(z,y,x) \equiv \underline{\mathcal{T}}(z,y,x)\ \underline{R}(y,x) \tag{17.6}$$

$$\underline{\mathcal{T}}(x,y,z) \equiv \underline{T}(x,y)[\underline{I}_m - \underline{R}(y,z)\ \underline{R}(y,x)]^{-1} \tag{17.7}$$

$$\underline{\mathcal{R}}(x,y,z) \equiv \underline{\mathcal{T}}(x,y,z)\ \underline{R}(y,z) \tag{17.8}$$

The *complete upward and downward source-generated irradiances* at level y are defined as

$$\underline{\mathcal{H}}_\eta(z,y,x) \equiv [\underline{H}_\eta(z,y) + \underline{H}_\eta(x,y)\ \underline{R}(y,z)][\underline{I}_m - \underline{R}(y,x)\ \underline{R}(y,z)]^{-1} \tag{17.9}$$

$$\underline{\mathcal{H}}_\eta(x,y,z) \equiv [\underline{H}_\eta(x,y) + \underline{H}_\eta(z,y)\ \underline{R}(y,x)][\underline{I}_m - \underline{R}(y,z)\ \underline{R}(y,x)]^{-1} \tag{17.10}$$

The reader should take note of the special magnitudes of $\underline{\mathit{R}}$, $\underline{\mathit{T}}$, and $\underline{H}_\eta$ when y takes on the end values x or z in its range $x \le y \le z$. For example, $\underline{\mathit{T}}(z,z,x) = \underline{I}_m$, $\underline{\mathit{T}}(z,x,x) = \underline{T}(z,x)$; moreover, $\underline{\mathit{R}}(z,z,x) = \underline{R}(z,x)$, $\underline{\mathit{R}}(z,x,x) = \underline{0}_m$; and finally, $\underline{\mathit{H}}_\eta(z,z,x) = 0$, $\underline{\mathit{H}}_\eta(z,x,x) = \underline{H}_\eta(z,x)$. These values follow from the assumption that X(x,z) is a bare slab.

The physical interpretations of the imbed rule are readily made. For example, H(y,+) in (17.3) is the linear combination of three terms: that due to the two incident irradiances $\underline{H}(z,+)$ and $\underline{H}(x,-)$, and a contribution from the internal source-generated irradiances in the layers just above and below level y. In particular, $\underline{\mathit{T}}(z,y,x)$ in (17.3) is the *complete transmittance* operator (an m×m matrix) which, on reading (17.5), is seen to be made up of an initial transmittance T(z,y) from level z to level y followed by an infinite interreflection upward between layers X(x,y) and X(y,z):

$$[\underline{I}_m - \underline{R}(y,x)\underline{R}(y,z)]^{-1} = \underline{I}_m + \underline{R}(y,x)\underline{R}(y,z) + [\underline{R}(y,x)\underline{R}(y,z)]^2 + \cdots \quad (17.11)$$

$\underline{\mathit{R}}(x,y,z)$ in (17.3) is the *complete reflectance* operator and is the result of a complete downward transmittance to level y (cf. (17.8)) which is followed by a reflection in X(y,z) at level y. The complete upward source-generated irradiance $\underline{\mathit{H}}_\eta(z,y,x)$ in (17.3) and (17.9) at level y is the sum of two terms: $\underline{H}_\eta(z,y)$ generated in X(y,z) which is infinitely interreflected upward at level y and $\underline{H}_\eta(x,y)$ generated in X(x,y) which is first reflected in X(y,z) and then infinitely interreflected upward, again in the manner of (17.11).

B. *Analytic Imbedding in* X(x,z)

The analytic approach to the imbed rule rests on the fact that the four $\underline{R}$, $\underline{T}$ matrices in (17.5)-(17.8) and the pair of source-generated functions $\underline{H}_\eta$ in (17.9), (17.10), as functions of y, obey the local interaction principles (11.19). Hence all six functions may be generated by numerical integration sweeps over X(x,z) starting from appropriate initial values.

To see that the $\underline{R}$, $\underline{T}$ and $\underline{H}_\eta$ functions satisfy (11.19), write (17.3) and (17.4) together as

$$\underline{H}(y) = (\underline{H}(z,+), \underline{H}(x,-)) \, \underline{M}(x,y,z) + \underline{H}_\eta(y) \tag{17.12}$$

where

$$\underline{M}(x,y,z) \equiv \begin{bmatrix} \underline{T}(z,y,x) & \underline{R}(z,y,x) \\ \underline{R}(x,y,z) & \underline{T}(x,y,z) \end{bmatrix} \tag{17.13}$$

and

$$\underline{H}_\eta(y) \equiv [\underline{H}_\eta(z,y,x), \underline{H}_\eta(x,y,z)] \tag{17.14}$$

Now we assume that $\underline{H}(y)$ in (17.12) is a solution of (11.19). Then from (11.19), which we repeat here,

$$\frac{d}{dy} \underline{H}(y) = \underline{H}(y) \, K(y) + \underline{h}_e(y) \tag{17.15a}$$

we have, by hypothesis (17.12)

$$\frac{d}{dy} \underline{H}(y) = [(H(z,+), H(x,-)) \, \underline{M}(x,y,z) + \underline{H}_\eta(y)] \, \underline{K}(y) + \underline{h}_e(y) \tag{17.15b}$$

Differentiating (17.12) with respect to y, and holding x,z fixed, we obtain

$$\frac{d}{dy}\,\underline{H}(y) = [H(z,+),\ H(x,-)]\,\frac{d}{dy}\,\underline{\mathcal{M}}(x,y,z) + \frac{d}{dy}\,\underline{H}_\eta(y) \tag{17.16}$$

By construction, $\underline{H}(z,+)$, $\underline{H}(y,-)$ and the emission sources generating $\underline{H}_\eta(y)$ are pairwise independent and arbitrary. Therefore we find, on comparing (17.15b) and (17.16), that necessarily

$$\boxed{\begin{aligned}\frac{d}{dy}\,\underline{\mathcal{M}}(x,y,z) &= \underline{\mathcal{M}}(x,y,z)\,\underline{\mathcal{K}}(y) \\ \frac{d}{dy}\,\underline{\mathcal{H}}_\eta(y) &= \underline{\mathcal{H}}_\eta(y)\,\underline{\mathcal{K}}(y) + \underline{h}_e(y)\end{aligned}} \qquad \begin{aligned}(17.17)\\(17.18)\end{aligned}$$

which are the local interaction forms for $\underline{\mathcal{M}}(x,y,z)$ and $\underline{\mathcal{H}}_\eta(y)$.

We can retrace our steps to (17.15a) by starting with (17.17) and (17.18), which define $\underline{\mathcal{M}}(x,y,z)$ and $\underline{\mathcal{H}}_\eta(y)$. In this way we see that (17.17) and (17.18) provide an equivalent solution procedure for (11.19). Some details for this reverse procedure follow: Construct $\underline{H}(y)$ via (17.12). Differentiate (17.12) to obtain (17.16). Simplify (17.16) via (17.17), (17.18) to obtain (17.15b) and then, once again by (17.12), we arrive at (17.15a).

What are the initial values for (17.17) and (17.18)? Consider, for example, the case of a bare slab X(x,z) (no interactive boundaries within it). The required initial values for $\underline{\mathcal{M}}(x,y,z)$ and $\underline{H}_\eta(y)$ are then given by (17.5)-(17.10):

$$\underline{\mathcal{M}}(x,x,z) = \begin{bmatrix} \underline{\mathcal{T}}(z,x,x) & \underline{\mathcal{R}}(z,x,x) \\ \underline{\mathcal{R}}(x,x,z) & \underline{\mathcal{T}}(x,x,z) \end{bmatrix} = \begin{bmatrix} T(z,x) & \underline{0}_m \\ \underline{R}(x,z) & \underline{I}_m \end{bmatrix} \tag{17.19}$$

and

$$\underline{M}(x,z,z) = \begin{bmatrix} \underline{T}(z,z,x) & \underline{R}(z,z,x) \\ \underline{R}(x,z,z) & \underline{T}(x,z,z) \end{bmatrix} = \begin{bmatrix} \underline{I}_m & \underline{R}(z,x) \\ \underline{0}_m & \underline{T}(x,z) \end{bmatrix} \tag{17.20}$$

and finally

$$\underline{H}_\eta(x) = [\underline{H}_\eta(z,x,x),\ \underline{H}_\eta(x,x,z)] = [\underline{H}_\eta(z,x),\ \underline{0}] \tag{17.21}$$

$$\underline{H}_\eta(z) = [\underline{H}_\eta(z,z,x),\ \underline{H}_\eta(x,z,z)] = [\underline{0},\ \underline{H}_\eta(x,z)] \tag{17.22}$$

These initial values suggest that we can integrate the system (17.17), (17.18) in either of two separate sweeps, one sweep upward from z to x or one downward from x to z. Now, writing out (17.17) in more detail,

$$\frac{d}{dy}\begin{bmatrix} \underline{T}(z,y,x) & \underline{R}(z,y,x) \\ \underline{R}(x,y,z) & \underline{T}(x,y,z) \end{bmatrix} = \begin{bmatrix} \underline{T}(z,y,x) & R(z,y,x) \\ \underline{R}(x,y,z) & \underline{T}(x,y,z) \end{bmatrix} \begin{bmatrix} -\underline{\tau}(y,+) & \underline{\rho}(y,+) \\ -\underline{\rho}(y,-) & \underline{\tau}(y,-) \end{bmatrix} \tag{17.23}$$

and reading off components of this matrix equation, we find

$$\left.\begin{aligned} -\frac{d}{dy}\underline{R}(x,y,z) &= \underline{R}(x,y,z)\ \underline{\tau}(y,+) + \underline{T}(x,y,z)\ \underline{\rho}(y,-) \\ \frac{d}{dy}\underline{T}(x,y,z) &= \underline{T}(x,y,z)\ \underline{\tau}(y,-) + \underline{R}(x,y,z)\ \underline{\rho}(y,+) \end{aligned}\right\} \tag{17.24}$$

and

$$
\left.\begin{aligned}
-\frac{d}{dy}\,\mathcal{T}(z,y,x) &= \underline{\mathcal{T}}(z,y,x)\ \underline{\tau}(y,+) + \underline{\mathcal{R}}(z,y,x)\ \underline{\rho}(y,-) \\
\frac{d}{dy}\,\mathcal{R}(z,y,x) &= \underline{\mathcal{R}}(z,y,x)\ \underline{\tau}(y,-) + \underline{\mathcal{T}}(z,y,x)\ \underline{\rho}(y,+)
\end{aligned}\right\}\qquad (17.25)
$$

Further, (17.18) yields

$$
\left.\begin{aligned}
-\frac{d}{dy}\,\underline{\mathcal{H}}_\eta(z,y,x) &= \underline{\mathcal{H}}_\eta(z,y,x)\ \underline{\tau}(y,+) + \underline{\mathcal{H}}_\eta(x,y,z)\ \underline{\rho}(y,-) + \underline{h}_e(y,+) \\
\frac{d}{dy}\,\underline{\mathcal{H}}_\eta(x,y,z) &= \underline{\mathcal{H}}_\eta(x,y,z)\ \underline{\tau}(y,-) + \underline{\mathcal{H}}_\eta(z,y,x)\ \underline{\rho}(y,+) + \underline{h}_e(y,-)
\end{aligned}\right\}\qquad (17.26)
$$

To continue our illustration of the use of these equations to analytically realize the imbed rule, suppose we integrate the system (17.24)-(17.26) upward from z to x over the bare slab X(x,z). By (17.20) and (17.22) the initial values of the $\underline{\mathcal{R}}$, $\underline{\mathcal{T}}$ and $\underline{\mathcal{H}}_\eta$ are

$$
\left.\begin{aligned}
\underline{\mathcal{R}}(x,z,z) &= \underline{0}_m \\
\underline{\mathcal{T}}(x,z,z) &= \underline{\mathcal{T}}(x,z) \\
\underline{\mathcal{T}}(z,z,x) &= \underline{I}_m \\
\underline{\mathcal{R}}(z,z,x) &= \underline{\mathcal{R}}(z,x) \\
\\
\underline{\mathcal{H}}_\eta(z,z,x) &= \underline{0} \\
\underline{\mathcal{H}}_\eta(x,z,z) &= \underline{H}_\eta(x,z)
\end{aligned}\right\}
\quad
\begin{array}{l}
\text{initial } \underline{\mathcal{R}},\ \underline{\mathcal{T}},\ \underline{\mathcal{H}}_\eta \text{ values} \\
\text{for upward sweep of system} \\
\text{(17.24)-(17.26) over bare} \\
\text{layer } X(x,z) \\
x \le y \le z
\end{array}
\qquad
\begin{array}{r}
(17.27) \\
\\
\\
(17.28)
\end{array}
$$

The three non-trivial starting values $\underline{\mathcal{R}}(z,x)$, $\underline{\mathcal{T}}(x,z)$, and $\underline{H}_\eta(x,z)$ are obtained by a preliminary downward sweep of the major Riccati trio (16.6)-(16.8) with bare-slab initial values (16.12). The upward integration of (17.24)-(17.26) may be stopped for inspection at any level y. At that level we would have the four $\underline{\mathcal{R}}$, $\underline{\mathcal{T}}$ matrices and the two vectors $\underline{\mathcal{H}}_\eta$ needed to calculate $(\underline{H}(y,+), \underline{H}(y,-))$ via (17.3)-(17.4). The integration may then continue upward or downward from level y to obtain $\underline{R}$, $\underline{T}$ matrices and the $\underline{H}_\eta$ vectors for further calculations of $\underline{H}(y)$ via (17.3)-(17.4).

C. *Algebraic Imbedding in* X(a,b) = X(a,y) U X(y,b)

We now consider the algebraic version of a very general (the *inductive*) form of the imbed rule (17.3), (17.4). We establish the imbed rule for the light field at level y between two general media X(a,y) and X(y,b) for each of which the $\underline{R}$, $\underline{T}$ and $\underline{H}_\eta$ quantities are *assumed* known and for each of which statements of the global interaction principles of sec. 15 are *assumed* to hold. From these assumptions, the present form of the imbed rule will be derived. The present procedure is therefore fundamentally different, logically speaking, from that in par. A. The derivation of the imbed rule in par. A serves to show that there is at least one non-empty case of application of the rule. This follows because the $\underline{R}$, $\underline{T}$ and $\underline{H}_\eta$ used there, along with the global interaction principles exist as mathematical entities and are determined from the local interaction principle (11.19) using an existence theorem for solutions of the differential equation system (11.19) or its equivalent form in (17.17), (17.18). The existence theorem can be applied, by construction, to bare slabs such as X(x,z) in par. A. Put in more practical terms, the constructions of par. A can actually be realized numerically using the steps described in par. B.

The derivation of the imbed rule now to be undertaken serves as an inductive step toward the general imbed rule. We shall show that if the global interaction principles hold on X(a,y) and X(y,b), then the imbed rule can be formulated for $\underline{H}(y,\pm)$. For this purpose, the natures of X(a,y) and X(y,b) are deliberately left unspecified; this is where the power of the present inductive step lies. For example, X(a,y) could be a zero-thickness air-water surface and X(y,b) the water below X(a,y) which includes the hydrosol bottom. Then $\underline{H}(y,\pm)$ given by (17.33) and (17.34), below will be the light field just under the air water surface. When the present general form of the imbed rule is used together with the general inductive form of the union rule of section 18C, we can build, in successive stages, transfer functions and their global interaction principles that apply to arbitrarily stratified media with any internal arrangement of bare slabs and plane surfaces.

Having made these preliminary observations, the remainder of the development of the inductive form of the imbed rule now proceeds quite simply, as follows. We start with the assumed global interaction principles representing the response irradiance fields on X(a,y) and X(y,b) in terms of the incident irradiances on them:

on X(a,y):

$$\underline{H}(a,+) = \underline{H}(y,+)\,\underline{T}(y,a) + \underline{H}(a,-)\,\underline{R}(a,y) + \underline{H}_\eta(y,a) \qquad (17.29)$$

$$\underline{H}(y,-) = \underline{H}(y,+)\,\underline{R}(y,a) + \underline{H}(a,-)\,\underline{T}(a,y) + \underline{H}_\eta(a,y) \qquad (17.30)$$

on X(y,b):

$$\underline{H}(y,+) = \underline{H}(b,+)\,\underline{T}(b,y) + \underline{H}(y,-)\,\underline{R}(y,b) + \underline{H}_\eta(b,y) \qquad (17.31)$$

$$\underline{H}(b,-) = \underline{H}(b,+)\,\underline{R}(b,y) + \underline{H}(y,-)\,\underline{T}(y,b) + \underline{H}_\eta(y,b) \qquad (17.32)$$

Using (17.30) and (17.31) we may solve for $\underline{H}(y,\pm)$ in terms of the incident irradiances $\underline{H}(a,-)$ and $\underline{H}(b,+)$ to find the general algebraic form of the *imbed rule*:

$$\boxed{\begin{aligned} \underline{H}(y,+) &= \underline{H}(b,+)\,\underline{\mathcal{T}}(b,y,a) + \underline{H}(a,-)\,\underline{\mathcal{R}}(a,y,b) + \underline{\mathcal{H}}_\eta(b,y,a) \\ \underline{H}(y,-) &= \underline{H}(b,+)\,\underline{\mathcal{R}}(b,y,a) + \underline{H}(a,-)\,\underline{\mathcal{T}}(a,y,b) + \underline{\mathcal{H}}_\eta(a,y,b) \end{aligned}} \qquad \begin{aligned}(17.33)\\(17.34)\end{aligned}$$

where the *complete reflectance* and *complete transmittance* m×m matrices are

$$\underline{\mathcal{T}}(b,y,a) \equiv \underline{T}(b,y)[\underline{I}_m - \underline{R}(y,a)\ \underline{R}(y,b)]^{-1} \tag{17.35}$$

$$\underline{\mathcal{R}}(b,y,a) \equiv \underline{\mathcal{T}}(b,y,a)\ \underline{R}(y,a) \tag{17.36}$$

$$\underline{\mathcal{T}}(a,y,b) \equiv \underline{T}(a,y)[\underline{I}_m - \underline{R}(y,b)\ \underline{R}(y,a)]^{-1} \tag{17.37}$$

$$\underline{\mathcal{R}}(a,y,b) \equiv \underline{\mathcal{T}}(a,y,b)\ \underline{R}(y,b) \tag{17.38}$$

a
y
b

And where the *complete upward and downward source-generated irradiances* at level y are defined as

$$\underline{\mathcal{H}}_\eta(b,y,a) \equiv [\underline{H}_\eta(b,y) + \underline{H}_\eta(a,y)\ \underline{R}(y,b)][\underline{I}_m - \underline{R}(y,a)\ \underline{R}(y,b)]^{-1} \tag{17.39}$$

$$\underline{\mathcal{H}}_\eta(a,y,b) \equiv [\underline{H}_\eta(a,y) + \underline{H}_\eta(b,y)\ \underline{R}(y,a)][\underline{I}_m - \underline{R}(y,b)\ \underline{R}(y,a)]^{-1} \tag{17.40}$$

The matrices $\underline{R}(a,a)$, $\underline{R}(b,b)$, by convention, are $\underline{0}_m$; while $\underline{T}(a,a)$, $\underline{T}(b,b)$ are $\underline{I}_m$.

D. *Analytic Imbedding in* X(a,b) = X(a,x) U X(x,z) U X(z,b)

We may extend the discussion in par. B to its general inductive form. Write (17.33) and (17.34) as

$$\underline{H}(y) = [\underline{H}(b,+),\ H(a,-)]\ \underline{\mathcal{M}}(a,y,b) + \underline{\mathcal{H}}_\eta(y) \tag{17.41}$$

where

$$\underline{\mathcal{M}}(a,y,b) \equiv \begin{bmatrix} \underline{\mathcal{T}}(b,y,a) & \underline{\mathcal{R}}(b,y,a) \\ \underline{\mathcal{R}}(a,y,b) & \underline{\mathcal{T}}(a,y,b) \end{bmatrix} \tag{17.42}$$

a
x
y
z
b

$$\underline{\mathcal{H}}_\eta(y) \equiv [\underline{\mathcal{H}}_\eta(b,y,a),\ \underline{\mathcal{H}}_\eta(a,y,b)] \tag{17.43}$$

and where y ranges over the interval $[x,z]$, $a \le x \le y \le z \le b$, in which it is assumed that $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$ and $\underline{h}_e(y,\pm)$ are continuous functions of y. We may then use the argument of par. B to derive the present versions of (17.17) and (17.18) (replace 'x' and 'z' by 'a' and 'b', respectively):

$$\boxed{\begin{aligned} &\frac{d}{dy}\underline{M}(a,y,b) = \underline{M}(a,y,b)\,\underline{K}(y) \qquad (17.44)\\ &\frac{d}{dy}\underline{H}_\eta(y) = \underline{H}_\eta(y)\,\underline{K}(y) + \underline{h}_e(y) \qquad (17.45)\\ &x \le y \le z \end{aligned}}$$

Since y ranges over only the internal layer $X(x,z)$, $x \le y \le z$, we must consider the boundary values of $\underline{M}(a,y,b)$ and $\underline{H}_\eta(y)$ for y at x and at z. However, first, for later reference, (17.44) and (17.45) are written out in the form of the *imbedding sextet*:

$$\left.\begin{aligned} -\frac{d}{dy}\underline{R}(a,y,b) &= \underline{R}(a,y,b)\,\underline{\tau}(y,+) + \underline{T}(a,y,b)\,\underline{\rho}(y,-)\\ \frac{d}{dy}\underline{T}(a,y,b) &= \underline{T}(a,y,b)\,\underline{\tau}(y,-) + \underline{R}(a,y,b)\,\underline{\rho}(y,+) \end{aligned}\right\} \qquad (17.44a,b)$$

and

$$\left.\begin{aligned} -\frac{d}{dy}\underline{T}(b,y,a) &= \underline{T}(b,y,a)\,\underline{\tau}(y,+) + \underline{R}(b,y,a)\,\underline{\rho}(y,-)\\ \frac{d}{dy}\underline{R}(b,y,a) &= \underline{R}(b,y,a)\,\underline{\tau}(y,-) + \underline{T}(b,y,a)\,\underline{\rho}(y,+) \end{aligned}\right\} \qquad (17.44c,d)$$

and finally

$$\left.\begin{aligned} -\frac{d}{dy}\,\underline{H}_\eta(b,y,a) &= \underline{H}_\eta(b,y,a)\,\underline{\tau}(y,+) + \underline{H}_\eta(a,y,b)\,\underline{\rho}(y,-) + \underline{h}_e(y,+) \\ \frac{d}{dy}\,\underline{H}_\eta(a,y,b) &+ \underline{H}_\eta(a,y,b)\,\underline{\tau}(y,-) + \underline{H}_\eta(b,y,a)\,\underline{\rho}(y,+) + \underline{h}_e(y,-) \end{aligned}\right\} \quad (17.45a,b)$$

The boundary values for $\underline{M}(a,y,b)$ and $\underline{H}_\eta(y)$ are

$$\underline{M}(a,x,b) = \begin{bmatrix} \underline{T}(b,x,a) & \underline{R}(b,x,a) \\ \underline{R}(a,x,b) & \underline{T}(a,x,b) \end{bmatrix} \qquad (17.46)$$

$$\underline{M}(a,z,b) = \begin{bmatrix} \underline{T}(b,z,a) & \underline{R}(b,z,a) \\ \underline{R}(a,z,b) & \underline{T}(a,z,b) \end{bmatrix} \qquad (17.47)$$

and finally

$$\underline{H}_\eta(x) = [\underline{H}_\eta(b,x,a),\ \underline{H}_\eta(a,x,b)] \qquad (17.48)$$

$$\underline{H}_\eta(z) = [\underline{H}_\eta(b,z,a),\ \underline{H}_\eta(a,z,b)] \qquad (17.49)$$

in which the components are given by the pair (17.39), (17.40) (with the substitutions $y \to x$ and $y \to z$, respectively, in each pair). The explicit expressions for these boundary values, $\underline{T}(b,x,a),\ldots,\underline{H}_\eta(a,z,b)$, in terms of the transfer functions of the boundaries X(a,x), X(z,b), and the media X(x,b), X(a,z) are given by (17.35)-(17.40) with the appropriate substitutions $y \to x$ or $y \to z$. Observe that, while X(a,x) is by hypothesis a boundary medium, X(x,b) is generally a composite medium of the form X(x,z) U X(z,b), as used in the formulations of the initial values in (17.46). Similar remarks hold for the boundary X(z,b) and the generally composite medium X(a,z) involved in (17.47). In practice the transfer functions and source-generated irradiances for all these media must be known before the present analytic form of the imbed rule can be used. This prior information is of course simply part of the inductive hypothesis of the general analytic and algebraic imbed rules now under construction.

We can now create the analytical counterpart to (17.33) and (17.34) by integrating (17.44) and (17.45) in an upward sweep, say from z to an arbitrary level y in [x,z]. The argument given in par. B, which started with (17.17) and (17.18) and led back to (17.15a), may be repeated in all its essential steps, now for (17.44) and (17.45). Hence $\underline{H}(y)$, constructed as in (17.33) and (17.34), satisfies (17.15a) over the range $x \le y \le z$, i.e., $\underline{H}(y)$ satisfies the basic local interaction principle (11.19).

It remains to show that the $\underline{H}(y,\pm)$, as given in (17.33) and (17.34), satisfy the following hypothesized boundary conditions on the boundaries X(a,x) and X(z,b) of X(a,b):

for X(a,x):

$$\underline{H}(a,+) = \underline{H}(x,+)\,\underline{T}(x,a) + \underline{H}(a,-)\,\underline{R}(a,x) + \underline{H}_\eta(x,a) \tag{17.50}$$

$$\underline{H}(x,-) = \underline{H}(x,+)\,\underline{R}(x,a) + \underline{H}(a,-)\,\underline{T}(a,x) + \underline{H}_\eta(a,x) \tag{17.51}$$

for X(z,b):

$$\underline{H}(z,+) = \underline{H}(b,+)\,\underline{T}(b,z) + \underline{H}(z,-)\,\underline{R}(z,b) + \underline{H}_\eta(b,z) \tag{17.52}$$

$$\underline{H}(b,-) = \underline{H}(b,+)\,\underline{R}(b,z) + \underline{H}(z,-)\,\underline{T}(z,b) + \underline{H}_\eta(z,b) \tag{17.53}$$

Here the trio $\underline{R}$, $\underline{T}$ and $\underline{H}_\eta$ for each direction are assumed given. The boundary value check on (17.33), (17.34) for the case of the upper boundary X(a,x) will now be made. Set y = x in (17.33) and (17.34) to find

$$\underline{H}(x,+) = \underline{H}(b,+)\ \underline{\mathcal{T}}(b,x,a) + \underline{H}(a,-)\ \mathcal{R}(a,x,b) + \underline{\mathcal{H}}_\eta(b,x,a) \tag{17.54}$$

$$\underline{H}(x,-) = \underline{H}(b,+)\ \underline{\mathcal{R}}(b,x,a) + \underline{H}(a,-)\ \mathcal{T}(a,x,b) + \underline{\mathcal{H}}_\eta(a,x,b) \tag{17.55}$$

We now introduce the expression for $\underline{H}(x,+)$, given in (17.54), into the right side of (17.51), and reduce the result. The boundary value check on (17.33), (17.34) is successful if the reduced expression is precisely the right side of (17.55). Here are the details:

$$\begin{aligned}
\underline{A} \equiv \left\{\begin{array}{l}\text{right side}\\ \text{of (17.51)}\\ \text{via right}\\ \text{side of}\\ \text{(17.54)}\end{array}\right\} &\equiv [\underline{H}(b,+)\ \underline{\mathcal{T}}(b,x,a) + \underline{H}(a,-)\ \underline{\mathcal{R}}(a,x,b) + \underline{\mathcal{H}}_\eta(b,x,a)]\ \underline{R}(x,a) \\
&\quad + \underline{H}(a,-)\ \underline{T}(a,x) + \underline{H}_\eta(a,x) \\
&= \underline{H}(b,+)\ [\underline{\mathcal{T}}(b,x,a)\ \underline{R}(x,a)] && (17.56a) \\
&\quad + \underline{H}(a,-)[\underline{T}(a,x) + \underline{\mathcal{R}}(a,x,b)\ \underline{R}(x,a)] && (17.56b) \\
&\quad + \underline{H}_\eta(a,x) + \underline{\mathcal{H}}_\eta(b,x,a)\ \underline{R}(x,a) && (17.56c)
\end{aligned}$$

Now observe from (17.36) that

$$\underline{\mathcal{T}}(b,x,a)\ \underline{\mathcal{R}}(x,a) = \underline{\mathcal{R}}(b,x,a) \tag{17.57a}$$

Moreover, from (17.38) and (17.37) observe that

$$\underline{T}(a,x) + \underline{\mathcal{R}}(a,x,b)\ \underline{R}(x,a)$$

$$= \underline{T}(a,x) + \underline{T}(a,x)[\underline{I}_m-\underline{R}(x,b)\ \underline{R}(x,a)]^{-1}\ \underline{R}(x,b)\ \underline{R}(x,a)$$

$$= \underline{T}(a,x)[\underline{I}_m + (\underline{I}_m-\underline{R}(x,b)\ \underline{R}(x,a))^{-1}\ \underline{R}(x,b)\ \underline{R}(x,a)]$$

$$= \underline{T}(a,x)[\underline{I}_m - \underline{R}(x,b)\ \underline{R}(x,a)]^{-1}$$

$$= \underline{\mathcal{T}}(a,x,b) \tag{17.57b}$$

Finally, from (17.39) and (17.40) observe that

$$\underline{H}_\eta(a,x) + \underline{\mathcal{H}}_\eta(b,x,a)\ \underline{R}(x,a)$$

$$= \underline{H}_\eta(a,x) + [\underline{H}_\eta(b,x) + \underline{H}_\eta(a,x)\ \underline{R}(x,b)][\underline{I}_m - \underline{R}(x,a)\ \underline{R}(x,b)]^{-1}\ \underline{R}(x,a)$$

$$= \underline{H}_\eta(a,x)\ [\underline{I}_m + \underline{R}(x,b)\ \underline{R}(x,a)(\underline{I}_m - \underline{R}(x,b)\ \underline{R}(x,a))^{-1}]$$

$$+ \underline{H}_\eta(b,x)\ \underline{R}(x,a)[\underline{I}_m - \underline{R}(x,b)\ \underline{R}(x,a)]^{-1}$$

$$= [H_\eta(a,x) + \underline{H}_\eta(b,x)\ \underline{R}(x,a)][\underline{I}_m - \underline{R}(x,b)\ \underline{R}(x,a)]^{-1}$$

$$= \underline{\mathcal{H}}_\eta(a,x,b) \tag{17.57c}$$

Using (17.57a,b,c) in (17.56a,b,c), respectively, we see that

$$\underline{A} = \underline{H}(b,+)\ \underline{\mathcal{R}}(b,x,a) + \underline{H}(a,-)\ \underline{\mathcal{T}}(a,x,b) + \underline{\mathcal{H}}_\eta(a,x,b) \tag{17.58}$$

which is the right side of (17.55), as was to be shown. The remaining check at the lower boundary X(z,b) is left to the reader: set y = z in (17.33) and (17.34). Take the resultant expression for $\underline{H}(z,-)$ and insert it into (17.52) and reduce the algebra in the manner illustrated in (17.57). The reduced result should be the expression for $\underline{H}(z,+)$ given by (17.33).

In summary, what we have shown is that the light field $\underline{\mathrm{H}}(y)$, *given by (17.33) and (17.34) (the algebraic imbedding rule), under the inductive hypothesis, satisfies (11.19) and the required boundary conditions (17.50)-(17.53). Moreover, we have shown that we can construct* $\underline{\mathrm{H}}(y)$ *on* $X[x,y]$ *in* $X[a,b]$ *by integration sweeps of (17.44), (17.45) (the analytic imbed rule) using the hypothesized boundary values (17.46)-(17.49).*

E. *Some Connections Among the Complete Operators*

We next summarize some connections among the complete operators $\underline{R}$, $\underline{T}$ and the complete irradiance $\underline{H}_\eta$ that hold in a general medium $X(a,b)$. Two of these connections were encountered as a matter of course in the form of (15.57b), (15.57c), while checking boundary conditions on (17.33) and (17.34). We now collect these relations together into their natural families as they occur in (17.33), (17.34). First, we have

$$\boxed{\begin{aligned} &\underline{R}(a,y,b) = \underline{T}(a,y,b)\ \underline{\mathrm{R}}(y,b) && (17.59)\\ &\underline{T}(b,y,a) = \underline{\mathrm{T}}(b,y) + \underline{R}(b,y,a)\ \underline{\mathrm{R}}(y,b) && (17.60)\\ &\underline{H}_\eta(b,y,a) = \underline{\mathrm{H}}_\eta(b,y) + \underline{H}_\eta(a,y,b)\ \underline{\mathrm{R}}(y,b) && (17.61)\end{aligned}}$$

a
y
b

Similarly, on interchanging a and b in the preceding relations,

$$\boxed{\begin{aligned} &\underline{R}(b,y,a) = \underline{T}(b,y,a)\ \underline{\mathrm{R}}(y,a) && (17.62)\\ &\underline{T}(a,y,b) = \underline{\mathrm{T}}(a,y) + \underline{R}(a,y,b)\ \underline{\mathrm{R}}(y,a) && (17.63)\\ &\underline{H}_\eta(a,y,b) = \underline{\mathrm{H}}_\eta(a,y) + \underline{H}_\eta(b,y,a)\ \underline{\mathrm{R}}(y,a) && (17.64)\end{aligned}}$$

a
y
b

The dual set to these connections will be derived in the section on the union rule (cf. (18.25)-(18.30)).

F. *Transport Trios for the Complete Operators* ($\underline{T}$, $\underline{H}_\eta$ *form)*

The final stage of preparation for solving the local interaction principles (11.19) will now be taken. What we shall do here is to transform the imbedding sextet (17.44), (17.45) into a pair of trios that specifically exhibit the stable character of a transport solution procedure of the local interaction principles. We first saw this stable character in (12.18)-(12.21) by virtue of the presence of the $\underline{K}(y,\pm)$ matrix function, where $\underline{K}(y,\pm)$ are in effect negative matrices at each level y that serve to *decrease* $\underline{H}(y,\pm)$ in the ± directions of photon flow. It is this decay of $\underline{H}(y,\pm)$ in the ± direction that is the root of the stability of the transport solution of (11.19). However the irradiances $\underline{H}(y,\pm)$ decrease along the ± directions only under suitable boundary conditions on them. Accordingly, the matter of suitable boundary conditions on the $\underline{H}(y,\pm)$ must finally be settled, and we shall do it here. It turns out that we can work on this matter most efficiently through the complete $\underline{R}$ and $\underline{T}$ operators and the complete irradiances $\underline{H}_\eta$ of the sextet (17.44)-(17.45), rather than on H(y,±) themselves.

Recall that we obtained the stable transport forms (12.18) and (12.21) of (11.19) through the introduction of the $\underline{R}(y,\pm)$ matrices, defined in (12.5) and (12.7). However the $\underline{R}(y,\pm)$ do not fully account for the boundary conditions on H(y,±); and some preliminary comments below (12.8) and below (12.11) on this point were made. The concept needed to handle this matter is the complete $\underline{R}$ operator of the imbed rule (17.33), (17.34): on the one hand it behaves like an irradiance with suitable boundary values; on the other, it has the character of a reflectance. Accordingly, what we will do below is to use $\underline{\mathcal{R}}(a,y,b)$ instead of $\underline{R}(y,-)$ to effect the transformation of (17.44)-(17.45) to their transport form. The details will now be considered.

Using (17.62) in (17.44b) and (17.64) in (17.45b), and rearranging the results, we obtain the *downward transport trio for* $\underline{\mathcal{T}}$, $\underline{\mathcal{H}}_\eta$ *and* $\underline{\mathcal{R}}$

$$\frac{d}{dy}\,\underline{\mathcal{T}}(a,y,b) = \underline{\mathcal{T}}(a,y,b)[\underline{\tau}(y,-) + \underline{R}(y,b)\ \underline{\rho}(y,+)] \qquad (17.65)$$

$$\frac{d}{dy}\,\underline{\mathcal{H}}_\eta(a,y,b) = \underline{\mathcal{H}}_\eta(a,y,b)[\underline{\tau}(y,-) + \underline{R}(y,b)\ \underline{\rho}(y,+)] + [\underline{H}_\eta(b,y)\ \underline{\rho}(y,+) + \underline{h}_e(y,-)] \qquad (17.66)$$

$$\underline{\mathcal{R}}(a,y,b) = \underline{\mathcal{T}}(a,y,b)\ \underline{R}(y,b) \qquad (17.67)$$

$$\begin{array}{l} a\ \rule{2cm}{0.4pt} \\ x\ \rule{2cm}{0.4pt} \\ y\ \rule{2cm}{0.4pt} \\ z\ \rule{2cm}{0.4pt} \\ b\ \rule{2cm}{0.4pt} \end{array}$$

$$a \le x \le y \le z \le b$$

For reference convenience we have repeated (17.62) in (17.67) to complete the present trio. Observe how the combination $\underline{\tau}(y,-) + \underline{R}(y,b)\ \underline{\rho}(y,+)$ occurs in both differential equations (17.65), (17.66) and compare it with $\underline{K}(y,-)$ in (12.19). We therefore have in $\underline{\tau}(y,-) + \underline{R}(y,b)\ \underline{\rho}(y,+)$ a decay function, now in matrix rather than numerical form (as it first occurred in the monochromatic irradiance theory). Hence $\underline{\mathcal{T}}(a,y,b)$ tends to decay exponentially as y increases. So does $\underline{\mathcal{H}}_\eta(a,y,b)$ tend to decay; but $\underline{\mathcal{H}}_\eta(a,y,b)$ also has two sources of growth: the true source density $\underline{h}_e(y,-)$ for the downward flux, and the locally reflected upward source-generated irradiance $\underline{H}_\eta(b,y)$. The presences of 'a' and 'b' in $\underline{\mathcal{H}}_\eta(a,y,b)$ and $\mathcal{R}(a,y,b)$ serve to remind us of the boundary conditions that $\underline{\mathcal{H}}_\eta$ and $\underline{\mathcal{R}}$ must satisfy. To start the downward integration of (17.65) and (17.66) just below the upper boundary X(a,x) of the medium X(a,b), we need the initial values $\underline{\mathcal{T}}(a,x,b)$ and $\underline{\mathcal{H}}_\eta(a,x,b)$ of $\underline{\mathcal{T}}$ and $\underline{\mathcal{H}}_\eta$. These are given by

$$\underline{\mathcal{T}}(a,x,b) = \underline{T}(a,x)[\underline{I}_m - \underline{R}(x,b)\ \underline{R}(x,a)]^{-1} \tag{17.68}$$

$$\underline{\mathcal{H}}_\eta(a,x,b) = [\underline{H}_\eta(a,x) + \underline{H}_\eta(b,x)\ \underline{R}(x,a)][\underline{I}_m - \underline{R}(x,b)\ \underline{R}(x,a)]^{-1} \tag{17.69}$$

The matrices $\underline{T}(a,x)$, $\underline{R}(x,a)$ and the vector $\underline{H}_\eta(a,x)$ are known from the given boundary conditions (17.50)-(17.53). Hence (17.65) and (17.66) can be integrated from x to z in the bare interior X(x,z) of X(a,b), providing we have $\underline{\tau}(y,-)$, $\underline{\rho}(y,+)$, $\underline{h}_e(y,-)$ and $\underline{R}(y,b)$ at each level y, $x \le y \le z$. These generally will be available in the integration procedures to be assembled and discussed in section 19.

The upward counterpart to (17.65)-(17.67) may be derived and discussed in a similar manner. Thus, use (17.59) in (17.44c), and (17.61) in (17.45a).

Rearranging the results, we obtain the *upward transport trio for* $\underline{\mathcal{T}}$, $\underline{\mathcal{H}}_\eta$ *and* $\underline{\mathcal{R}}$:

$$-\frac{d}{dy}\,\underline{\mathcal{T}}(b,y,a) = \underline{\mathcal{T}}(b,y,a)[\underline{\tau}(y,+) + \underline{R}(y,a)\ \underline{\rho}(y,-)] \tag{17.70}$$

$$-\frac{d}{dy}\,\underline{\mathcal{H}}_\eta(b,y,a) = \underline{\mathcal{H}}_\eta(b,y,a)[\underline{\tau}(y,+) + \underline{R}(y,a)\ \underline{\rho}(y,-)] + [\underline{H}_\eta(a,y)\ \underline{\rho}(y,-) + \underline{h}_e(y,+)] \tag{17.71}$$

$$\underline{\mathcal{R}}(b,y,a) = \underline{\mathcal{T}}(b,y,a)\ \underline{R}(y,a) \tag{17.72}$$

$$a \le x \le y \le z \le b$$

a ———
x ———
y ———
z ———
b ———

The associated boundary conditions are

$$\underline{\mathcal{T}}(b,z,a) = \underline{T}(b,z)[\underline{I}_m - \underline{R}(z,a)\,\underline{R}(z,b)]^{-1} \tag{17.73}$$

$$\underline{\mathcal{H}}_\eta(b,z,a) = [\underline{H}_\eta(b,z) + \underline{H}_\eta(a,z)\,\underline{R}(z,b)][\underline{I}_m - \underline{R}(z,a)\,\underline{R}(z,b)]^{-1} \tag{17.74}$$

G. *Transport Trios for the Complete Operators ($\underline{\mathcal{R}}$,$\underline{\mathcal{H}}_\eta$ form)*

The transport trios of the $\underline{\mathcal{T}}$, $\underline{\mathcal{H}}_\eta$ form derived in par. F have dual counterparts in the form of the ($\underline{\mathcal{R}}$,$\underline{\mathcal{H}}_\eta$) pairs, and we shall summarize these dual trios here. We start again with the imbedding sextet (17.44)-(17.45). Using (17.60) in (17.44a) and (17.61) in (17.45a) we find, after rearrangements, the *downward transport trio for* $\underline{\mathcal{R}}$, $\underline{\mathcal{H}}_\eta$ *and* $\underline{\mathcal{T}}$:

$$-\frac{d}{dy}\underline{\mathcal{R}}(a,y,b) = \underline{\mathcal{R}}(a,y,b)[\underline{\tau}(y,+) + \underline{R}(y,a)\,\underline{\rho}(y,-)] + \underline{T}(a,y)\,\underline{\rho}(y,-) \tag{17.75}$$

$$\begin{aligned}-\frac{d}{dy}\underline{\mathcal{H}}_\eta(b,y,a) &= \underline{\mathcal{H}}_\eta(b,y,a)[\underline{\tau}(y,+) + \underline{R}(y,a)\,\underline{\rho}(y,-)] \\ &\quad + [\underline{H}_\eta(a,y)\,\underline{\rho}(y,-) + \underline{h}_e(y,+)]\end{aligned} \tag{17.76}$$

$$\underline{\mathcal{T}}(a,y,b) = \underline{\mathcal{T}}(a,y) + \underline{\mathcal{R}}(a,y,b)\,\underline{R}(y,a) \tag{17.77}$$

$$a \le x \le y \le z \le b$$

$$\begin{array}{l} a \text{ ———} \\ x \text{ ———} \\ y \text{ ———} \\ z \text{ ———} \\ b \text{ ———} \end{array}$$

The initial values are

$$\underline{\mathcal{R}}(a,x,b) = \underline{T}(a,x)\,\underline{R}(x,b)[\underline{I}_m - \underline{R}(x,a)\,\underline{R}(x,b)]^{-1} \tag{17.78}$$

$$\underline{\mathcal{H}}_\eta(b,x,a) = [\underline{H}_\eta(b,x) + \underline{H}_\eta(a,x)\,\underline{R}(x,b)][\underline{I}_m - \underline{R}(x,a)\,\underline{R}(x,b)]^{-1} \tag{17.79}$$

Observe how both equations (17.75) and (17.76) show the same decay matrix $\underline{\tau}(y,+) + \underline{R}(y,a)\,\underline{\rho}(y,-)$ for upward-flowing photons (compare with (12.20)). Yet the direction of integration for this pair is downward (-) into X(a,b), i.e., in the direction of increasing y. This actually assures, under the given boundary conditions, the decay of $\underline{\mathit{R}}(a,y,b)$ and $\underline{\mathit{H}}(b,y,a)$ in the (-) direction: This is corroborated by the presence of $\underline{\mathit{T}}(a,y,b)$ in (17.77) which, as we saw in the $\underline{\mathit{T}}$, $\underline{\mathit{H}}_\eta$ trio (17.65)-(17.67), decays in the downward direction, i.e., as y increases. Also notice in (17.76) that the two source terms are being *deintegrated*, i.e., their natural additions to the growth of $\underline{\mathit{H}}_\eta(b,y,a)$ (which is normally growing in the upward direction) are being continuously subtracted in the downward march from x to z. We have here, therefore, some interesting scattering dynamics; could (17.75) and (17.76) (or (17.77), for that matter) ever have been written down on purely intuitive grounds?

Finally, the *upward transport trio for* $\underline{\mathit{R}}$, $\underline{\mathit{H}}_\eta$ *and* $\underline{\mathit{T}}$ may be written down at once from (17.75) on interchanging 'a' and 'b' along with '+' and '-':

$$\frac{d}{dy}\,\underline{\mathit{R}}(b,y,a) = \underline{\mathit{R}}(b,y,a)[\underline{\tau}(y,-) + \underline{\mathrm{R}}(y,b)\;\underline{\rho}(y,+)] + \underline{\mathrm{T}}(b,y)\;\underline{\rho}(y,+) \tag{17.80}$$

$$\frac{d}{dy}\,\underline{\mathit{H}}_\eta(a,y,b) = \underline{\mathit{H}}_\eta(a,y,b)[\underline{\tau}(y,-) + \underline{\mathrm{R}}(y,b)\;\underline{\rho}(y,+)] + [\underline{\mathrm{H}}_\eta(b,y)\;\underline{\rho}(y,+) + \underline{\mathrm{h}}_e(y,-)] \tag{17.81}$$

$$\underline{\mathit{T}}(b,y,a) = \underline{\mathit{T}}(b,y) + \underline{\mathit{R}}(b,y,a)\;\underline{\mathrm{R}}(y,b) \tag{17.82}$$

a
x
y
z
b

The initial values are

$$\underline{\mathit{R}}(b,z,a) = \underline{\mathrm{T}}(b,z)\;\underline{\mathrm{R}}(z,a)[\underline{\mathrm{I}}_m - \underline{\mathrm{R}}(z,b)\;\underline{\mathrm{R}}(z,a)]^{-1} \tag{17.83}$$

$$\underline{\mathit{H}}_\eta(a,z,b) = [\underline{\mathrm{H}}_\eta(a,z) + \underline{\mathrm{H}}_\eta(b,z)\;\underline{\mathrm{R}}(z,a)][\underline{\mathrm{I}}_m - \underline{\mathrm{R}}(z,b)\;\underline{\mathrm{R}}(z,a)]^{-1} \tag{17.84}$$

The integration of the two trios in their respective directions can proceed once the initial conditions are at hand and the local reflectances and transmittances $\rho(y,\pm)$, $\tau(y,\pm)$ are known along with local source terms $\underline{h}_e(y,\pm)$ within $X(x,z)$, and also the global structures $R(y,a)$, $R(y,b)$, $\underline{H}_\eta(a,y)$, $\underline{H}_\eta(b,y)$. The latter are supplied by ancillary integrations of the Riccati trios of section 16. These matters will be discussed further in the solution procedure assembled in section 19.

H. *The Link Between* $\underline{M}(a,y,b)$, $\underline{M}(a,x,b)$ *and* $\underline{M}(x,y)$

The $2m \times 2m$ matrix operator $\underline{M}(a,y,b)$ may be linked to $\underline{M}(a,x,b)$ by means of the fundamental matrix $\underline{M}(x,y)$, as follows. This link will show us how the complete operators R, T change with depth via the fundamental matrix $\underline{M}$. By (17.41), on the one hand we have for arbitrary incident irradiances $[H(b,+),H(a,-)]$ and sources within $X(a,b)$, the field at depth y given by

$$[\underline{H}(y,+),\ \underline{H}(y,-)] = [\underline{H}(b,+),\ \underline{H}(a,-)]\ \underline{M}(a,y,b) + \underline{H}_\eta(y) \tag{17.85}$$

and in particular

$$[\underline{H}(x,+),\ \underline{H}(x,-)] = [\underline{H}(b,+),\ \underline{H}(a,-)]\ \underline{M}(a,x,b) + \underline{H}_\eta(x) \tag{17.86}$$

On the other hand, by (14.18), we have

$$[\underline{H}(y,+),\ \underline{H}(y,-)] = [\underline{H}(x,+),\ \underline{H}(x,-)]\ \underline{M}(x,y) + \underline{H}_e(x,y) \tag{17.87}$$

This suggests applying $\underline{M}(x,y)$ to each side of (17.86) to obtain

$$[\underline{H}(b,+),\ \underline{H}(a,-)]\ \underline{M}(a,x,b)\ \underline{M}(x,y) + \underline{H}_\eta(x)\ \underline{M}(x,y) \tag{17.88a}$$

$$= [\underline{H}(x,+),\ \underline{H}(x,-)]\ \underline{M}(x,y) \tag{17.88b}$$

$$= [\underline{H}(y,+),\ \underline{H}(y,-)] - \underline{H}_e(x,y) \tag{17.88c}$$

$$= [\underline{H}(b,+),\ \underline{H}(a,-)]\ \underline{M}(a,y,b) + \underline{H}_\eta(y) - \underline{H}_e(x,y) \tag{17.88d}$$

The step from (17.88b) to (17.88c) used (17.87); the transition from (17.88c) to (17.88d) used (17.85). Since $\underline{H}(b,+)$, $\underline{H}(a,-)$ and $\underline{h}_e(y,\pm)$, $x \le y \le y$, are arbitrary, we find

$$\boxed{\begin{aligned} &\underline{M}(a,y,b) = \underline{M}(a,x,b)\ \underline{M}(x,y) \\ &\underline{H}_\eta(y) = \underline{H}_\eta(x)\ \underline{M}(x,y) + \underline{H}_e(x,y) \\ &x \le y \le z \end{aligned}} \qquad \begin{aligned} &(17.89a) \\ &(17.89b) \end{aligned}$$

which are the desired connections. Observe the similarity of (17.89b) to (14.43). Moreover, we see that the link between $\underline{M}(a,x,b)$ and $\underline{M}(a,y,b)$ is the fundamental matrix $\underline{M}(x,y)$. It is instructive to open up (17.89a) into its component equations:

$$\begin{bmatrix} \underline{T}(b,y,a) & \underline{R}(b,y,a) \\ \underline{R}(a,y,b) & \underline{T}(a,y,b) \end{bmatrix} = \begin{bmatrix} \underline{T}(b,x,a) & \underline{R}(b,x,a) \\ \underline{R}(a,x,b) & \underline{T}(a,x,b) \end{bmatrix} \begin{bmatrix} \underline{M}_{++}(x,y) & \underline{M}_{+-}(x,y) \\ \underline{M}_{-+}(x,y) & \underline{M}_{--}(x,y) \end{bmatrix}$$

That is,

$$\underline{\mathcal{R}}(a,y,b) = \underline{\mathcal{R}}(a,x,b)\ \underline{\mathcal{M}}_{++}(x,y) + \underline{\mathcal{T}}(a,x,b)\ \underline{\mathcal{M}}_{-+}(x,y) \tag{17.90a}$$

$$\underline{\mathcal{T}}(a,y,b) = \underline{\mathcal{R}}(a,x,b)\ \underline{\mathcal{M}}_{+-}(x,y) + \underline{\mathcal{T}}(a,x,b)\ \underline{\mathcal{M}}_{--}(x,y) \tag{17.90b}$$

$$\underline{\mathcal{T}}(b,y,a) = \underline{\mathcal{T}}(b,x,a)\ \underline{\mathcal{M}}_{++}(x,y) + \underline{\mathcal{R}}(b,x,a)\ \underline{\mathcal{M}}_{-+}(x,y) \tag{17.90c}$$

$$\underline{\mathcal{R}}(b,y,a) = \underline{\mathcal{T}}(b,x,a)\ \underline{\mathcal{M}}_{+-}(x,y) + \underline{\mathcal{R}}(b,x,a)\ \underline{\mathcal{M}}_{--}(x,y) \tag{17.90d}$$

A useful special case of (17.90) occurs in bare homogeneous medium. $X(x,z)$, i.e., where $X(x,z)$ has no reflecting boundaries and $\underline{\rho}_{\pm},\underline{\tau}_{\pm}$ are independent of depth. Then (17.90) become

$$\underline{\mathcal{R}}(x,y,z) = \underline{R}(x,z)\ \underline{\mathcal{M}}_{++}(x,y) + \underline{\mathcal{M}}_{-+}(x,y) \tag{17.91a}$$

$$\underline{\mathcal{T}}(x,y,z) = \underline{R}(x,z)\ \underline{\mathcal{M}}_{+-}(x,y) + \underline{\mathcal{M}}_{--}(x,y) \tag{17.91b}$$

$$\underline{\mathcal{T}}(z,y,x) = \underline{T}(z,x)\ \underline{\mathcal{M}}_{++}(x,y) \tag{17.91c}$$

$$\underline{\mathcal{R}}(z,y,x) = \underline{T}(z,x)\ \underline{\mathcal{M}}_{+-}(x,y) \tag{17.91d}$$

For computations using these forms, by homogeneity we may replace $\underline{\mathcal{M}}_{++}(x,y),\ldots,\underline{\mathcal{M}}_{--}(x,y)$ by their eigenmatrix representations in (13.42). Observe that the end values of $\underline{\mathcal{R}}(x,y,z),\ldots,\underline{\mathcal{R}}(z,y,x)$ in (17.92) at $y = x$ or $y = z$, have the correct form. For example, in (17.91a), let $y \to x$, then

$$\begin{aligned}\underline{\mathcal{R}}(x,x,z) &= \underline{R}(x,z)\ \underline{\mathcal{M}}_{++}(x,x) + \underline{\mathcal{M}}_{-+}(x,x)\\ &= \underline{R}(x,z)\end{aligned}$$

which follows from (14.6), and checks with the extreme value of the representation derivable from (17.8). The representations (17.91) are particularly helpful in seeing the form of the y-dependence of the complete operators (via the eigenmatrix expressions (13.42a-d) for $\mathcal{M}_{++}(x,y)$, etc.) and also the boundary values of these operators (via the presence of the $\underline{\mathcal{R}}(x,z)$ and $\underline{T}(z,x)$ matrices). This interpretation also holds more generally for (17.90) when $X(a,b) = X(a,x) \cup X(x,z) \cup X(z,b)$ and when $X(x,z)$ is bare and homogeneous. Expressions (13.42a-d) can be inserted into (17.90a-d) to obtain very useful *analytical* representations of the complete $\underline{\mathcal{R}}$ and $\underline{\mathcal{T}}$ operators.

I. $\underline{\mathcal{M}}(a,y,b)$ *in Terms of* $\underline{\mathcal{M}}(y,a)$ *and* $\underline{\mathcal{M}}(y,b)$

It is possible to represent the complete operators of $\underline{\mathcal{M}}(a,y,b)$, along with the source-induced irradiances, solely in terms of the fundamental matrices $\underline{\mathcal{M}}(y,a)$ and $\underline{\mathcal{M}}(y,b)$. This may be done as follows. On the one hand the light field at level y in $X(a,b) = X(a,x) \cup X(x,z) \cup X(z,b)$, $x \le y \le z$, is given via (17.41) by

$$[\underline{H}(y,+),\ \underline{H}(y,-)] = [\underline{H}(b,+),\ \underline{H}(a,-)]\ \underline{\mathcal{M}}(a,y,b) + \underline{\mathcal{H}}_{\eta}(y) \qquad (17.92a)$$

On the other hand the light fields on levels a and b are related to that on level y by means of (14.18):

$$[\underline{H}(b,+),\underline{0}] = [\underline{H}(y,+),\ \underline{H}(y,-)]\ \underline{\mathcal{M}}(y,b)\ \underline{C}_{+} + \underline{H}_{e}(y,b)\ \underline{C}_{+} \qquad (17.92b)$$

$$[0,\ \underline{H}(a,-)] = [\underline{H}(y,+),\ \underline{H}(y,-)]\ \underline{\mathcal{M}}(y,a)\ \underline{C}_{-} + \underline{H}_{e}(y,a)\ \underline{C}_{-} \qquad (17.92c)$$

where $\underline{C}_{\pm}$ are the contraction operators introduced in (15.4). Here we have used the general properties of $\underline{C}_{\pm}$:

$$
\begin{aligned}
[\underline{A},\underline{B}]\underline{C}_+ &= [\underline{A},\underline{B}] \begin{bmatrix} \underline{I}_m & \underline{O}_m \\ \underline{O}_m & \underline{O}_m \end{bmatrix} = [\underline{A},\underline{O}] \\
[\underline{A},\underline{B}]\underline{C}_- &= [\underline{A},\underline{B}] \begin{bmatrix} \underline{O}_m & \underline{O}_m \\ \underline{O}_m & \underline{I}_m \end{bmatrix} = [\underline{O},\underline{B}]
\end{aligned}
\tag{17.93}
$$

where $\underline{A},\underline{B}$ are both either m×m matrices or 1×m vectors. Adding (17.92b,c) and using (17.92a), we obtain

$$
[\underline{H}(b,+),\ \underline{H}(a,-)] = \{[\underline{H}(b,+),\ \underline{H}(a,-)]\ \underline{\mathcal{M}}(a,y,b) + \underline{H}_\eta(y)\}[\underline{\mathcal{M}}(y,b)\ \underline{C}_+ + \underline{\mathcal{M}}(y,a)\ \underline{C}_-]
$$

$$
+ [\underline{H}_e(y,b)\ \underline{C}_+ + \underline{H}_e(y,a)\ \underline{C}_-]
$$

On rearranging this we have

$$
[\underline{H}(b,+),\ \underline{H}(a,-)]\{\underline{I}_m - \underline{\mathcal{M}}(a,y,b)[\underline{\mathcal{M}}(y,b)\ \underline{C}_+ + \underline{\mathcal{M}}(y,a)\ \underline{C}_-]\}
$$

$$
= \underline{\mathcal{H}}_\eta(y)[\underline{\mathcal{M}}(y,b)\ \underline{C}_+ + \underline{\mathcal{M}}(y,a)\ \underline{C}_-] + [\underline{H}_e(y,b)\ \underline{C}_+ + \underline{H}_e(y,a)\ \underline{C}_-]
$$

Since the incident irradiances $\underline{H}(b,+)$, $\underline{H}(a,-)$ are arbitrary and independent of internal sources, we find

$$
\boxed{
\begin{aligned}
&\underline{\mathcal{M}}(a,y,b) = [\underline{\mathcal{M}}(y,b)\ \underline{C}_+ + \underline{\mathcal{M}}(y,a)\ \underline{C}_-]^{-1} && \text{(17.94a)} \\
&\underline{\mathcal{H}}_\eta(y) = -[\underline{H}_e(y,b)\ \underline{C}_+ + \underline{H}_e(y,a)\ \underline{C}_-][\underline{\mathcal{M}}(y,b)\ \underline{C}_+ + \underline{\mathcal{M}}(y,a)\ \underline{C}_-]^{-1} && \text{(17.94b)}
\end{aligned}
}
$$

Equation (17.94a) is the desired connection between $\underline{\mathcal{M}}(a,y,b)$ and the fundamental matrices $\underline{\mathcal{M}}(y,b)$, $\underline{\mathcal{M}}(y,a)$. We also obtained the link between the complete source-generated irradiance $\underline{\mathcal{H}}_\eta(y)$ and the fundamental source-generated irradiances $\underline{H}_e(y,b)$, $\underline{H}_e(y,a)$. This representation of $\underline{H}_\eta(y)$ should be compared with (17.9). One should also recall the origin of the transport version of the source irradiances $\underline{H}_\eta$ in (15.18).

It is instructive to open up (17.94a):

$$\begin{bmatrix} \underline{T}(b,y,a) & \underline{R}(b,y,a) \\ \underline{R}(a,y,b) & \underline{T}(a,y,b) \end{bmatrix} = \begin{bmatrix} \underline{M}_{++}(y,b) & \underline{M}_{+-}(y,a) \\ \underline{M}_{-+}(y,b) & \underline{M}_{--}(y,a) \end{bmatrix}^{-1} \tag{17.95}$$

whence, via (15.14),

$$\boxed{\begin{aligned} \underline{T}(b,y,a) &= [\underline{M}_{++}(y,b) - \underline{M}_{+-}(y,a)\, \underline{M}_{--}^{-1}(y,a)\, \underline{M}_{-+}(y,b)]^{-1} && (17.96a) \\ \underline{T}(a,y,b) &= [\underline{M}_{--}(y,a) - \underline{M}_{-+}(y,b)\, \underline{M}_{++}^{-1}(y,b)\, \underline{M}_{+-}(y,a)]^{-1} && (17.96b) \\ \underline{R}(a,y,b) &= -\underline{M}_{--}^{-1}(y,a)\, \underline{M}_{-+}(y,b)\, \underline{T}(b,y,a) && (17.96c) \\ \underline{R}(b,y,a) &= -\underline{M}_{++}^{-1}(y,b)\, M_{+-}(y,a)\, \underline{T}(a,y,b) && (17.96d) \end{aligned}}$$

Here we have written the formulas so that only $\underline{M}_{++}(y,b)$ and $\underline{M}_{--}(y,a)$ need be inverted. By using the other formulas in (15.14), other forms of representation are possible.

These formulas reduce correctly to all their known special cases. For instance, setting y = a in (17.96a), we have

$$\begin{aligned} \underline{T}(b,a,a) &= M_{++}^{-1}(a,b) \\ &= \underline{T}(b,a) \end{aligned}$$

Here we used the fact that $\underline{M}_{++}(a,a) = \underline{I}_m$ and $\underline{M}_{+-}(a,a) = \underline{0}_m$ (cf. (14.6)) and we also used the form for $\underline{T}(b,a)$ in terms of $M_{++}^{-1}(a,b)$ (cf. (15.17)). That $\underline{T}(b,a,a)$ should reduce to $\underline{T}(b,a)$ follows from the definition (17.35).

The forms (17.96) are particularly useful in homogeneous media $X(x,z)$ in which $\underline{M}(x,y)$ has its eigenmatrix representation (13.42). Then they may be written out in terms of elementary functions (exponentials). When there is no fluorescence, the present theory reduces to M uncoupled equations of the form (14.1), and (17.96) reduce to the case of $m = 1$, the scalar case. It is particularly instructive to study this simple case to see the stability feature of transport theory (i.e., that in $X(a,\infty)$, $\underline{T}(a,y,\infty)$ decreases exponentially and that $\underline{R}(a,y,\infty)$ goes to a *finite* limit as $y\to\infty$). For a discussion of this stability feature in a simple setting, see Preisendorfer (1977a, p. 40).

18. UNION RULE

The union rule leads to the determination of the global transfer functions $\underline{R}$, $\underline{T}$ and the source-generated irradiances $\underline{H}_\eta$ of the union of two water layers when one knows the optical properties of the two parts of the union. The general setting of the present constructions is that defined at the outset of section 17. Thus we consider, once again for didactive reasons, both the union $X(x,z)$ of bare slabs $X(x,y)$ and $X(y,z)$ and the union $X(a,b)$ of possibly composite slabs $X(a,y)$, $X(y,b)$. Our main goal is the inductive rule in par. C, below. We shall develop both the algebraic and the analytic forms of the union rule.

A. *Algebraic Union of* $X(x,z) = X(x,y) \cup X(y,z)$

Assume we have a bare slab $X(x,z)$ partitioned into a union of two contiguous bare slabs $X(x,y)$ and $X(y,z)$. Thus each part $X(x,y)$ and $X(y,z)$ has a quartet of $\underline{R}$ and $\underline{T}$ transfer matrices and a pair of source generated irradiances $\underline{H}_\eta$ that are known. The arbitrary pair of incident irradiances on each slab $X(x,y)$ and $X(y,z)$ combines under a global interaction principle to yield the associated pair of response irradiances, as demonstrated in (17.1) and (17.2).

Hence, by the imbed rule (17.3), (17.4), the irradiances $\underline{H}(y,\pm)$ at level y are known. We now use $\underline{H}(y,+)$, so found, in the first of the global interaction statements (15.20), and reformulate the latter as

$$\underline{H}(x,+) = H(y,+)\ T(y,x) + \underline{H}(x,-)\ \underline{R}(x,y) + \underline{H}_{\eta}(y,x)$$

$$= [H(z,+)\ \underline{\mathit{T}}(z,y,x) + \underline{H}(x,-)\ \underline{\mathit{R}}(x,y,z) + \underline{\mathit{H}}_{\eta}(z,y,x)]\ \underline{T}(y,x)$$

$$+ H(x,-)\ \underline{R}(x,y) + \underline{H}_{\eta}(y,x) \tag{18.1}$$

Collecting coefficients of the incident irradiances $\underline{H}(x,-)$ and $\underline{H}(z,+)$ on $X(x,y)$ as in (18.1), and collecting also the source-generated terms, we find

$$\underline{H}(x,+) = \underline{H}(z,+)[\underline{\mathit{T}}(z,y,x)\ \underline{T}(y,x)]$$

$$+ \underline{H}(x,-)[\underline{R}(x,y) + \underline{\mathit{R}}(x,y,z)\ \underline{T}(y,x)]$$

$$+ \underline{H}_{\eta}(y,x) + \underline{\mathit{H}}_{\eta}(z,y,x)\ \underline{T}(y,x) \tag{18.2}$$

Now since $X(x,z)$ is a bare slab, the theory of section 15 assigns a unique quartet of transfer matrices $\underline{T}(z,x)$, $\underline{\mathit{R}}(x,z)$, $\underline{T}(x,z)$, $\underline{R}(z,x)$ to $X(x,z)$ along with a unique pair $\underline{H}_{\eta}(z,x)$, $\underline{H}_{\eta}(x,z)$ of source-generated irradiances such that

$$\underline{H}(x,+) = \underline{H}(z,+)\ \underline{T}(z,x) + \underline{H}(x,-)\ \underline{R}(x,z) + \underline{H}_{\eta}(z,x) \tag{18.3}$$

$$\underline{H}(z,-) = \underline{H}(z,+)\ \underline{R}(z,x) + \underline{H}(x,-)\ \underline{T}(x,z) + \underline{H}_{\eta}(x,z) \tag{18.4}$$

On comparing (18.2) and (18.3) and noting the arbitrary magnitudes of the incident irradiances $\underline{H}(z,+)$, $\underline{H}(x,-)$ and of the emission vectors $\underline{h}_e(s,\pm)$, $x \le s \le z$, we conclude that

$$\underline{T}(z,x) = \underline{\mathit{T}}(z,y,x)\ \underline{T}(y,x) \tag{18.5}$$

$$\underline{R}(x,z) = \underline{R}(x,y) + \underline{\mathit{R}}(x,y,z)\ \underline{T}(y,x) \tag{18.6}$$

$$\underline{H}_\eta(z,x) = \underline{H}_\eta(y,x) + \underline{\mathit{H}}_\eta(z,y,x)\ \underline{T}(y,x) \tag{18.7}$$

x
y
z

Repeating the above argument, now with the known $\underline{H}(y,-)$ from (15.20) used in the second of the global interaction statements (15.36), and on comparison of the rearranged result (analogous to (18.2)) with (18.4), we arrive at

$$\underline{T}(x,z) = \underline{\mathit{T}}(x,y,z)\ \underline{T}(y,x) \tag{18.8}$$

$$\underline{R}(z,x) = \underline{R}(z,y) + \underline{\mathit{R}}(z,y,x)\ \underline{T}(y,z) \tag{18.9}$$

$$\underline{H}_\eta(x,z) = \underline{H}_\eta(y,z) + \underline{\mathit{H}}_\eta(x,y,z)\ \underline{T}(y,z) \tag{18.10}$$

x
y
z

Definitions of the complete operators $\underline{\mathit{R}}$, $\underline{\mathit{T}}$, and complete source-generated irradiances $\underline{\mathit{H}}_\eta$ are given in (17.35)-(17.40). The set of statements (18.5)-(18.10) constitute the *union rule* for contiguous bare slabs X(x,y) U X(y,z) = X(x,z).

B. *Analytic Union of* X(x,z) = X(x,y) U X(y,z)

The present analytic version of the union rule stems from the observation that a bare slab X(x,z) can be visualized as having been grown from a thinner bare slab X(y,z), $x < y < z$, by successively adding infinitesimal layers X(u,y) on top of X(y,z) so as eventually to arrive at X(x,z). At every intermediate stage of such a growth, we have a bare slab of the form X(u,z) = X(u,y) U X(y,z) where $x \le u \le y$. At this stage, the transfer functions of the union X(u,z) are given by (18.5)-(18.10) (with x replaced by u). For example, the trio (18.5)-(18.7) becomes

$$\underline{T}(z,u) = \underline{\mathcal{T}}(z,y,u)\ \underline{T}(y,u) \tag{18.11}$$

$$\underline{R}(u,z) = \underline{R}(u,y) + \underline{\mathcal{R}}(u,y,z)\ \underline{T}(y,u) \tag{18.12}$$

$$\underline{H}_\eta(z,u) = \underline{H}_\eta(y,u) + \underline{\mathcal{H}}_\eta(z,y,u)\ \underline{T}(y,u) \tag{18.13}$$

Now it is intuitively clear that $\underline{T}(z,u)$, $\underline{R}(u,z)$, and $\underline{H}_\eta(z,u)$, as given by (18.11)-(18.13), approach $\underline{T}(z,y)$, $\underline{R}(y,z)$, $\underline{H}_\eta(z,y)$ as u approaches y. Call the operators in (18.11)-(18.13) the *initial values* $\underline{T}(z,u)$, $\underline{R}(u,z)$, and $\underline{H}_\eta(z,u)$. It is less intuitively clear, but still possible to rigorously show, that $\underline{T}(z,u)$, $\underline{R}(u,z)$, and $\underline{H}_\eta(z,u)$, as the indicated algebraic combinations (18.11)-(18.13), $x \le u \le y$, satisfy the major Riccati trio (16.14)-(16.16) with substitution $y \to u$. Then, on integrating this Riccati trio upward from y to x with the above initial values (of y starting at u), we must recover the same $\underline{T}(z,x)$, $\underline{R}(x,z)$ and $\underline{H}_\eta(z,x)$ of $X(x,z)$ as given by the union rule (18.5)-(18.7). Symbolically realizing these simple insights produces the desired analytic form of the union rule.

We will now verify that $\underline{T}(z,u)$, $\underline{R}(u,z)$, and $\underline{H}_\eta(z,u)$, as given by (18.11)-(18.13), satisfy the major Riccati trio (16.14)-(16.16). Consider, for example, $\underline{T}(z,u)$. Differentiating each side of (18.11) with respect to u we have

$$\frac{\partial \underline{T}(z,u)}{\partial u} = \frac{\partial \underline{\mathcal{T}}(z,y,u)}{\partial u}\ \underline{T}(y,u) + \underline{\mathcal{T}}(z,y,u)\ \frac{\partial \underline{T}(y,u)}{\partial u} \tag{18.14}$$

By direct computation from the definition (17.35) of $\underline{\mathcal{T}}(z,y,u)$ and the Riccati sextets in section 16, we have*

$$- \frac{\partial \underline{\mathcal{T}}(z,y,u)}{\partial u} = \underline{T}(z,u)\ \underline{\rho}(u,+)\ \underline{\mathcal{R}}(u,y,z) \tag{18.15}$$

and also, by (16.15),

$$- \frac{\partial \underline{T}(y,u)}{\partial u} = \underline{T}(y,u)\ [\underline{\tau}(u,+) + \underline{\rho}(u,+)\ \underline{R}(u,y)] \tag{18.16}$$

Then with these observations, (18.14) becomes

$$- \frac{\partial \underline{T}(z,u)}{\partial u} = \underline{T}(z,u)\underline{\rho}(u,+)\underline{\mathcal{R}}(u,y,z)\underline{T}(y,u) + \underline{\mathcal{T}}(z,y,u)[\underline{T}(y,u)(\underline{\tau}(u,+) + \underline{\rho}(u,+)\underline{R}(u,y))] \tag{18.17}$$

Observe from (18.5) that

$$\underline{T}(z,u) = \underline{\mathcal{T}}(z,y,u)\ \underline{T}(y,u) \tag{18.18}$$

With this, (18.17) can be written,

$$- \frac{\partial \underline{T}(z,u)}{\partial u} = \underline{T}(z,u)[\underline{\rho}(u,+)(\underline{R}(u,y) + \underline{\mathcal{R}}(u,y,z)\ \underline{T}(y,u)) + \underline{\tau}(u,+)] \tag{18.19}$$

which by (18.6) becomes

$$- \frac{\partial \underline{T}(z,u)}{\partial u} = \underline{T}(z,u)[\underline{\tau}(u,+) + \underline{\rho}(u,+)\ \underline{R}(u,z)] \tag{18.20}$$

Thus $\underline{T}(z,u)$, as given in (18.11), obeys (16.15) of the Riccati trio, as was to

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* Alternately one can consult a list of such derivatives of $\underline{\mathcal{R}}$ and $\underline{\mathcal{T}}$; see Preisendorfer (1976, vol. IV, p. 75).

be shown. Therefore T(z,x) for the union X(x,z) = X(x,y) U X(y,z) can be obtained by either the algebraic or analytic form of the union rule.

That $\underline{R}(u,z)$ and $\underline{H}_\eta(z,u)$, as given by (18.12) and (18.13), satisfy (16.14) and (16.16), can be demonstrated in a similar way. This and the similar task for (18.8)-(18.10) are left to the reader.

C. *Algebraic Union of* X(a,b) = X(a,y) U X(y,b)

The present version of the union rule is its general inductive form, and therefore may be used with the inductive form of the Algebraic Imbedding rule of section 17C to form a powerful pair of construction procedures to solve (11.19) under a great variety of boundary value problems of both the usual external kind (as in (11.20)) and the less usual internal kind (to be defined in (v), (vi), of par. D, below).

Let us assume that an arbitrary layer of hydrosol X(a,b) is the union of two contiguous layers X(a,y) and X(y,b) for each of which the $\underline{R}$, $\underline{T}$, and $\underline{H}_\eta$ quantities are known and for each of which the global interaction principles (17.29)-(17.32) are assumed to hold. Therefore (17.33) and (17.34) yield up the imbedded light field $\underline{H}(y,\pm)$. Using (17.33) in (17.29) we obtain, on rearranging terms,

$$H(a,+) = [\underline{H}(b,+)\ \underline{\mathcal{T}}(b,y,a) + \underline{H}(a,-)\ \underline{\mathcal{R}}(a,y,b) + \underline{H}_\eta(b,y,a)]\ \underline{T}(y,a) + \underline{H}(a,-)\ \underline{R}(a,y) + \underline{H}_\eta(y,a) \tag{18.21}$$

$$\begin{aligned} &= \underline{H}(b,+)[\underline{\mathcal{T}}(b,y,a)\ \underline{T}(y,a)] \\ &+ \underline{H}(a,-)[\underline{R}(a,y) + \underline{\mathcal{R}}(a,y,b)\ \underline{T}(y,a)] \\ &+ \underline{H}_\eta(y,a) + \underline{\mathcal{H}}_\eta(b,y,a)\ \underline{T}(y,a) \end{aligned} \tag{18.22}$$

Now, from the general interaction principle that underlies all of classical radiative transfer theory*, to the optical medium X(a,b) with its external incident irradiances $\underline{H}(a,-)$, $\underline{H}(b,+)$, its internal incident irradiances $\underline{h}_e(s,\pm)$, $a \le s \le b$ and its two response irradiances $\underline{H}(a,+)$, $\underline{H}(b,-)$, we may assign four matrices $\underline{T}(b,a)$, $\underline{R}(b,a)$, $\underline{T}(a,b)$, $\underline{R}(a,b)$ and two source-generated fields $\underline{H}_\eta(a,+)$, $\underline{H}_\eta(b,-)$ such that

$$\underline{H}(a,+) = \underline{H}(b,+)\ \underline{T}(b,a) + \underline{H}(a,-)\ \underline{R}(a,b) + \underline{H}_\eta(b,a) \tag{18.23}$$

$$\underline{H}(b,-) = \underline{H}(b,+)\ \underline{R}(b,a) + \underline{H}(a,-)\ \underline{T}(a,b) + \underline{H}_\eta(a,b) \tag{18.24}$$

On comparing coefficients in (18.22) and (18.23), we find

$$\underline{T}(b,a) = \underline{\mathcal{T}}(b,y,a)\ \underline{T}(y,a) \tag{18.25}$$

(union rule, upward case)

$$\underline{R}(a,b) = \underline{R}(a,y) + \underline{\mathcal{R}}(a,y,b)\ \underline{T}(y,a) \tag{18.26}$$

a ———
y ———
b ———

$$\underline{H}_\eta(b,a) = \underline{H}_\eta(y,a) + \underline{\mathcal{H}}_\eta(b,y,a)\ \underline{T}(y,a) \tag{18.27}$$

Similarly, using (17.34) in (17.32), rearranging, and comparing with (18.24), we find

$$\underline{T}(a,b) = \underline{\mathcal{T}}(a,y,b)\ \underline{T}(y,b) \tag{18.28}$$

(union rule, downward case)

$$\underline{R}(b,a) = \underline{R}(b,y) + \underline{\mathcal{R}}(b,y,a)\ \underline{T}(y,b) \tag{18.29}$$

a ———
y ———
b ———

$$\underline{H}_\eta(a,b) = \underline{H}_\eta(y,b) + \underline{\mathcal{H}}_\eta(a,y,b)\ \underline{T}(y,b) \tag{18.30}$$

---

* Preisendorfer (1965, p. 114) or H.O. vol. II, p. 205.

The definitions of the complete operators and irradiances $\underline{\mathcal{R}}$, $\underline{\mathcal{T}}$ and $\underline{\mathcal{H}}_\eta$ are given in (17.35)-(17.40). Observe the rather interesting dual equation-structures between the above relations and those encountered in the imbedding section; cf. in particular (17.59)-(17.64).

D. *Building-Up Principle*

The *building-up principle* is a general procedure whereby the $\underline{R}$, $\underline{T}$ quartet and the source-generated irradiances $\underline{H}_\eta$ of a composite medium X(a,b) can be found by a sequence of applications of the union rule to the component parts of X(a,b). We consider some examples.

(i) Air-water surface X(a,x) and bare slab X(x,z) combined to form X(a,z) = X(a,x) U X(a,z).

(ii) Composite slab X(a,z) of (i) and bottom boundary surface X(z,b) combined to form X(a,b) = X(a,z) U X(z,b).

The result of the building-up principle applied to steps (i) and (ii) is the quartet of matrices $\underline{T}$(b,a), $\underline{R}$(b,a), $\underline{T}$(a,b), $\underline{R}$(a,b) and pair of source-generated irradiances $\underline{\mathcal{H}}_\eta$(a,b), $\underline{\mathcal{H}}_\eta$(b,a) for the composite medium X(a,b). Observe that at the end of stage (i) and just prior to performing (ii) we have the $\underline{R}$, $\underline{T}$, $\underline{\mathcal{H}}_\eta$ sextets for X(a,z) and X(z,b) available. At this point, before performing any further constructions, one may use the imbed rule to find $\underline{H}$(z,±). We will use this (union rule + imbed rule) tactic to solve the general boundary value problem for (11.19) in section 19.

Another useful pair of applications of the union rule is to build up X(a,b) as follows:

(iii) Bare water slab X(x,z) and bottom boundary surface X(z,b) combined to form X(x,b) = X(x,z) U X(z,b)

(iv) Air-water surface X(a,x) and composite slab X(x,b) of (i) combined to form X(a,b) = X(a,x) U X(x,b).

The resultant $\underline{R}$, $\underline{T}$, $\underline{H}_\eta$ sextet of step (iii) followed by (iv) should be identical to that produced by step (i) followed by (ii) for the medium X(a,b). Observe that, at the end of step (iii) and just before step (iv), one may use the imbed rule to find $\underline{H}(x,\pm)$. The net results of the steps (i)-(iv) combined with the imbed rule are that we are able to determine the response irradiance $\underline{H}(a,+)$, at the upper boundary of X(a,b), the response irradiance $\underline{H}(b,-)$, at the lower boundary of X(a,b) and the imbedded irradiances $\underline{H}(x,\pm)$ and $\underline{H}(z,\pm)$ at levels x,z just inside the boundaries of X(a,b).

A final pair of examples of the building up principle will be given which will illustrate the notion of an internal 'boundary' and the procedure to handle it in radiative transfer calculations. Suppose that there is a lake or sea with a sharp temperature discontinuity somewhere below the surface. At this surface interreflections may occur and it is of interest to estimate their general radiometric effect. (See, e.g., H.O., vol. I, p. 37.) Thus suppose the media $X(x,y_-)$, $X(y_+,z)$, $y_- \le y_+$, are bare media just above and just below the temperature discontinuity at depth y. We can estimate, using Fresnel reflectance theory, the quartet $\underline{r}(y_-,y_+)$, $\underline{t}(y_-,y_+)$, $\underline{r}(y_+,y_-)$, $\underline{t}(y_+,y_-)$ of the interface $X(y_-,y_+)$ between $X(x,y_-)$ and $X(y_+,z)$. Now consider

(v) Bare water slab $X(x,y_-)$ and interface $X(y_-,y_+)$ combined to form $X(x,y_+) = X(x,y_-) \cup X(y_-,y_+)$

(vi) Composite slab $X(x,y_+)$ and bare slab $X(y_+,z)$ combined to form

$$X(x,z) = X(x,y_+) \cup X(y_+,z)$$

Application of the union rule in (v) yields the quartet $R(x,y_+)$, $\underline{T}(x,y_+)$, $\underline{R}(y_+,x)$, $\underline{T}(y_+,x)$ and the pair $\underline{H}_\eta(x,y_+)$, $\underline{H}_\eta(y_+,x)$ in which we assume that $\underline{H}_\eta(y_-,y_+) = \underline{H}_\eta(y_+,y_-) = \underline{0}$ (no sources in the essentially zero thickness interface $X(y_-,y_+)$). Hence $\underline{H}_\eta(x,y_+)$ and $\underline{H}_\eta(y_+,x)$ are nonzero only if either $\underline{H}_\eta(x,y_-)$ or $\underline{H}_\eta(y_-,x)$, or both, are not zero. Just after stage (v) and before stage (vi) one can use the imbed rule to find $\underline{H}(y_+,\pm)$ just below the interface. The net result of (v) and (vi) is the $\underline{R}$, $\underline{T}$, $\underline{H}_\eta$ sextet for the composite layer $X(x,z)$ with internal interface $X(y_-,y_+)$ at depth y, $x < y < z$.

E. *Fundamental Matrix Products and Transport Star Products*

The building-up principle using the union rule can be summarized algebraically in terms of a kind of matrix product operation. We now explain this possibility, as it will lead to some rather useful facts about the interconnections between the fundamental and transport formulations of radiative transfer theory.

We return to (18.25)-(18.30) and recall how the four transfer matrices of $X(a,y)$ and of $X(y,b)$ have been combined to form the four transfer matrices of $X(a,b)$. Thus, for example, from

$$\underline{M}(a,y) = \begin{bmatrix} \underline{T}(y,a) & \underline{R}(y,a) \\ \underline{R}(a,y) & \underline{T}(a,y) \end{bmatrix} \tag{18.31}$$

and

$$\underline{M}(y,b) = \begin{bmatrix} \underline{T}(b,y) & \underline{R}(b,y) \\ \underline{R}(y,b) & \underline{T}(y,b) \end{bmatrix} \tag{18.32}$$

we find

$$\underline{M}(a,b) = \begin{bmatrix} \underline{T}(b,a) & \underline{R}(b,a) \\ \underline{R}(a,b) & \underline{T}(a,b) \end{bmatrix} \tag{18.33}$$

$$= \begin{bmatrix} \underline{\mathcal{T}}(b,y,a)\ \underline{T}(y,a) & \underline{R}(b,y) + \underline{\mathcal{R}}(b,y,a)\ \underline{T}(y,b) \\ \underline{R}(a,y) + \underline{\mathcal{R}}(a,y,b)\ \underline{T}(y,a) & \underline{\mathcal{T}}(a,y,b)\ \underline{T}(y,b) \end{bmatrix} \tag{18.34}$$

We summarize this binary relation (i.e., this combining, or union rule) by writing

$$'\underline{M}(a,y) * \underline{M}(y,b)' \quad \text{for} \quad \underline{M}(a,b) \tag{18.35}$$

We call $\underline{M}(a,y) * \underline{M}(y,b)$ the *star product* of $\underline{M}(a,y)$ and $\underline{M}(y,b)$. This star product is the algebraic essence of the union rule and hence of the building-up principle. (See H.O., vol. I, p. 46.)

The star product (18.35) has no special algorithmic value beyond (18.34). It does, however, have a noteworthy conceptual feature: It focuses attention on the algebraic structure of the union rule, and an important immediate consequence of the union rule. This is the generalization of the group properties (14.10)-(14.12) of $\underline{M}(u,v)$ to *composite* media (something which cannot automatically be done with fundamental matrices because of possible discontinuities in $\rho(y,\pm)$, $\tau(y,\pm)$ and $h_e(y,\pm)$. We now indicate how this can be done.

We have seen above how the union rule allows us to construct $\underline{M}(a,b)$ from $\underline{M}(a,y)$ and $\underline{M}(y,b)$ in a way that is analogous (but not identical) to forming a matrix product $\underline{\mathit{M}}(a,y)\ \underline{\mathit{M}}(y,b)$ of two fundamental matrices. We saw in (14.10) how the matrix product of $\underline{\mathit{M}}(u,v)$ and $\mathit{M}(v,w)$ allows us to propagate a solution of the local interaction differential equations from one level to another in a medium, provided that the local optical properties vary continuously with depth down through those levels. The star product, on the other hand, is an algebraic construct and so has no such analytic limitations. Indeed, the star product allows us to combine layers X(a,y) and X(y,b) of a medium X(a,b) with arbitrary internal structure (recall the illustrations of the building-up principle in par. D). We can in a similar manner endow the matrix product $\underline{\mathit{M}}(a,y)\ \underline{\mathit{M}}(y,b)$ of $\underline{\mathit{M}}(a,y)$ and $\underline{\mathit{M}}(y,b)$ with correct physical properties, and particularly with the property that the product is indeed $\underline{\mathit{M}}(a,b)$, by using this general discontinuity-bypassing feature of the star product $\underline{M}(a,y) * \underline{M}(y,b)$.

For this purpose we use the transformations introduced in section 15 between the fundamental ($\underline{\mathit{M}}$) and transport ($\underline{M}$) matrices. For example, (15.17) defines a mapping (i.e., a rule of transformation) from $\underline{\mathit{M}}(x,y)$ to $\underline{M}(x,y)$. Let us summarize this mapping by the function ϕ such that

$$\underline{M}(x,y) = \phi[\underline{\mathit{M}}(x,y)] \tag{18.36}$$

This mapping has a well-defined inverse ϕ^{-1} given by (15.26):

$$\underline{\mathit{M}}(x,y) = \phi^{-1}[\underline{M}(x,y)] \tag{18.37}$$

Suppose next that $X(a,b) = X(a,y) \cup X(y,b)$ is a composite medium consisting possibly of bare media bounded and separated by infinitesimally thin reflecting boundaries and reflecting interfaces across which the local transfer functions $\underline{\rho},\underline{\tau}$ have discontinuities. Then $\underline{\mathcal{M}}(a,b)$ cannot be generated from (14.4) in an uninterrupted integration sweep. For, if we numerically integrate across the discontinuity, its reflection activity on the surrounding light field can be missed. We now set up an inductive argument, much as in the building-up principle of par. D, above, for fundamental matrices that will allow them to handle such singular flux-reflection activity. Let the transfer matrices $\underline{M}(a,y)$ and $\underline{M}(y,b)$ for $X(a,y)$ and $X(y,b)$ be known. Then by (18.37) construct

$$\underline{\mathcal{M}}(a,y) \equiv \phi^{-1}[\underline{M}(a,y)] \tag{18.38a}$$

$$\underline{\mathcal{M}}(y,b) \equiv \phi^{-1}[\underline{M}(y,b)] \tag{18.38b}$$

and

$$\underline{\mathcal{M}}(a,b) \equiv \phi^{-1}[\underline{M}(a,b)] \ (= \phi^{-1}[\underline{M}(a,y) * \underline{M}(y,b)]) \tag{18.38c}$$

Now let an arbitrary incident irradiance $\underline{H}(a,-)$ generate an irradiance field throughout $X(a,b) = X(a,y) \cup X(y,b)$. If $\underline{H}(a,+)$, $\underline{H}(y,\pm)$, and $H(b,\pm)$ are the resultant fields in $X(a,b)$ we have, by construction of the $\underline{M}$-matrices, and hence their ϕ^{-1} images $\underline{\mathcal{M}}$, the three statements

$$[\underline{\mathrm{H}}(y,+),\ \underline{\mathrm{H}}(y,-)] = [\underline{\mathrm{H}}(a,+),\ \underline{\mathrm{H}}(a,-)]\ \underline{M}(a,y) \tag{18.39a}$$

$$[\underline{\mathrm{H}}(b,+),\ \underline{\mathrm{H}}(b,-)] = [\underline{\mathrm{H}}(y,+),\ \underline{\mathrm{H}}(y,-)]\ \underline{M}(y,b) \tag{18.39b}$$

and

$$[\underline{\mathrm{H}}(b,+),\ \underline{\mathrm{H}}(b,-)] = [\underline{\mathrm{H}}(a,+),\ \underline{\mathrm{H}}(a,-)]\ \underline{M}(a,b) \tag{18.39c}$$

Using (18.39a) in (18.39b), comparing the result with (18.39c), and recalling that $\mathrm{H}(a,-)$ (and hence $\mathrm{H}(a,+)$) is arbitrary, we find

$$\boxed{\underline{M}(a,b) = \underline{M}(a,y)\ \underline{M}(y,b)} \tag{18.40}$$

which establishes that $\underline{M}(a,b)$ is an ordinary matrix product of $\underline{M}(a,y)$ and $\underline{M}(y,b)$. This corresponds exactly to the star product

$$\boxed{\underline{\mathrm{M}}(a,b) = \underline{\mathrm{M}}(a,y) * \underline{\mathrm{M}}(y,b)} \tag{18.41}$$

by virtue of (18.38c). Moreover, substituting the definitions of the $\underline{M}$ matrices in (18.38), we can write (18.40) and (18.41) respectively as

$$\phi^{-1}[\underline{\mathrm{M}}(a,b)] = \phi^{-1}[\underline{\mathrm{M}}(a,b) * \underline{\mathrm{M}}(y,b)] = \phi^{-1}[\underline{\mathrm{M}}(a,y)]\ \phi^{-1}[\underline{\mathrm{M}}(y,b)] \tag{18.42a}$$

$$\phi[\underline{M}(a,b)] = \phi[\underline{M}(a,y)\ \underline{M}(y,b)] = \phi[\underline{M}(a,y)] * \phi[\underline{M}(y,b)] \tag{18.42b}$$

In this way we see the star product as the isomorphic correspondent* to matrix multiplication, and conversely. In this sense the $\underline{M}$ and $\underline{\mathrm{M}}$ formalisms

* For the notion of an isomorphism in modern algebra, see, e.g., Birkhoff and Maclane (1953).

are algebraically equivalent. However, this equivalence is one that holds only in an average sense, over the whole development of the theory. That is, our development of the $\underline{M}$ and $\underline{\mathrm{M}}$ formalisms, starting with section 15, shows that they are interestingly intertwined and symbiotic: in the present work, $\underline{M}(x,y)$ was developed first and then used to give the rigorous definition of $\underline{\mathrm{M}}(x,y)$ and lead us toward the global interaction principles on bare media; however, later, in the development of theory, $\underline{\mathrm{M}}(a,b)$ was used to give meaning to $\underline{M}(a,b)$ on composite media, as was just done. By keeping both these conceptual tools handy, we can solve any conceivable problem of radiative transfer in composite optical media, as will next be demonstrated.

19. TRANSPORT SOLUTION OF THE LIGHT FIELD: $\underline{R}\ \underline{T}\ \underline{H}_\eta$ ALGEBRAIC PROCEDURE

We now formulate the first of three solution procedures of (11.19) subject to various boundary conditions such as (11.20), and generalizations of (11.20) that include finitely thick fluorescing layers X(a,x), X(z,b) as boundaries. In the present section we first set down the general boundary data and boundary conditions on (11.19) that all three solution procedures will use. Then we will present the $\underline{R}\ \underline{T}\ \underline{H}_\eta$ *algebraic procedure*. In section 20 we present the $\underline{T}\ \underline{H}_\eta$ *analytic procedure* and in section 21, the $\underline{R}\ \underline{H}_\eta$ *analytic procedure*.

The $\underline{R}\ \underline{T}\ \underline{H}_\eta$ algebraic procedure is based on the algebraic form of the imbed rule presented in section 17C, while the $\underline{T}\ \underline{H}_\eta$ and $\underline{R}\ \underline{H}_\eta$ procedures stem from the analytic form of the imbed rule given in section 17D and as elaborated in pars. E and F of section 17.

A. *Given Data and Boundary Conditions (for the $\underline{R}\ \underline{T}\ \underline{H}_\eta$, $\underline{T}\ \underline{H}_\eta$, and $\underline{R}\ \underline{H}_\eta$ Finitely-Deep-Media Procedures)*

The present setting is a natural or laboratory stratified hydrosol X(a,b) of the form X(a,b) = X(a,x) U X(x,z) U X(z,b) where X(a,x) and X(z,b) are the *upper and lower boundaries* of X(a,b) while X(x,z) is its *interior*. X(x,z) is assumed to be a bare slab (cf. section 17). The boundary media can be of finite or infinite thickness and can have depth-distributed internal sources that produce emerging source-generated irradiances $\underline{H}_\eta$ at their upper and lower surfaces. The interior slab X(x,z) as usual may have internal sources of flux and by definition has continuously varying local properties. The interior is partitioned into n contiguous subslabs such that

$$X(x,z) = X(y_0,y_1) \cup \cdots \cup X(y_{j-1},y_j) \cup \cdots \cup X(y_{n-1},y_n) \qquad (19.1)$$

where $y_0 = x$ and $y_n = z$. It is at the levels y_j, $j = 0,\ldots,n$ that we want to determine the irradiances $H(y_j,\pm)$. Moreover, we wish to find the emerging irradiances $H(a,+)$, $H(b,-)$ at the upper and lower boundary surfaces. The given data and required irradiances are listed as follows:

Data Given:

(i) Boundary Data, $X(a,x)$:
$$\begin{cases} \text{a)} & \underline{R}(x,a),\ \underline{T}(a,x),\ \underline{H}_\eta(a,x) \\ \text{b)} & \underline{R}(a,x),\ \underline{T}(x,a),\ \underline{H}_\eta(x,a) \end{cases}$$

(ii) Boundary Data, $X(z,b)$:
$$\begin{cases} \text{a)} & \underline{R}(z,b),\ \underline{T}(b,z),\ \underline{H}_\eta(b,z) \\ \text{b)} & \underline{R}(b,z),\ \underline{T}(z,b),\ \underline{H}_\eta(z,b) \end{cases}$$

(iii)

Interior Data, $X(x,z)$:
$$\Big\{ \text{a)}\ \ \underline{\rho}(y,\pm),\ \underline{\tau}(y,\pm),\ \underline{h}_e(y,\pm),\ x \le y \le z$$

Incident Data:
$$\begin{cases} \text{b)} & \underline{H}(a,-) \\ \text{c)} & \underline{H}(b,+) \end{cases} \qquad \text{(incident irradiances)}$$

(iv) Boundary Conditions
$$\begin{cases} X(a,x): \begin{cases} \text{a)} & \underline{H}(a,+) = \underline{H}(x,+)\ \underline{T}(x,a) + \underline{H}(a,-)\ \underline{R}(a,x) + \underline{H}_\eta(x,a) \\ \text{b)} & \underline{H}(x,-) = \underline{H}(x,+)\ \underline{R}(x,a) + \underline{H}(a,-)\ \underline{T}(a,x) + \underline{H}_\eta(a,x) \end{cases} \\ X(z,b): \begin{cases} \text{c)} & \underline{H}(z,+) = \underline{H}(b,+)\ \underline{T}(b,z) + \underline{H}(z,-)\ \underline{R}(z,b) + \underline{H}_\eta(b,z) \\ \text{d)} & \underline{H}(b,-) = \underline{H}(b,+)\ \underline{R}(b,z) + \underline{H}(z,-)\ \underline{T}(z,b) + \underline{H}_\eta(z,b) \end{cases} \end{cases}$$

<u>Results Required</u>:

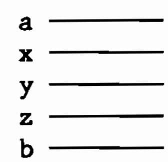
$$\begin{cases} \text{a) } \underline{H}(y_j,\pm),\ j = 0,\ldots,n \\ \text{b) } \underline{H}(a,+) \\ \text{c) } \underline{H}(b,-) \end{cases}$$

B. *The $\underline{R}$ $\underline{T}$ $\underline{H}_\eta$ Algebraic Procedure (Finitely Deep Media)*

There are six main stages of the $\underline{R}$ $\underline{T}$ $\underline{H}_\eta$ procedure:

I. Integrate the major Riccati trio (16.6)-(16.8) in the form

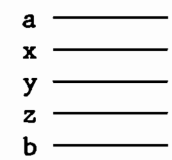
$$\begin{cases} \dfrac{\partial \underline{R}(y,a)}{\partial y} = \underline{R}(y,a)[\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\ \underline{R}(y,a)] \\[2ex] \dfrac{\partial \underline{T}(a,y)}{\partial y} = \underline{T}(a,y)[\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] \\[2ex] \dfrac{\partial \underline{H}_\eta(a,y)}{\partial y} = \underline{H}_\eta(a,y)[\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] + [\underline{h}_e(y,-) + \underline{h}_e(y,+)\ \underline{R}(y,a)] \end{cases}$$



The integration starts at level x with initial values (ia) (of par. A, above) and proceeds downward to level z using (iiia). Along the way the m×m matrices $\underline{R}(y_j,a)$, $\underline{T}(a,y_j)$ are stored with the 1×m vectors $\underline{H}_\eta(a,y_j)$, $j = 0,\ldots,n$. Here $\{y_j\colon\ j = 0,\ldots,n\}$ is the partition of X(x,z) decided upon in (19.1).

II. Integrate the major Riccati trio (16.14)-(16.16) in the form

$$\begin{cases} -\dfrac{\partial \mathrm{R}(y,b)}{\partial y} = \underline{\mathrm{R}}(y,b)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{\mathrm{R}}(y,b)] + [\underline{\rho}(y,-) + \underline{\tau}(y,-)\,\underline{\mathrm{R}}(y,b)] \\[2ex] -\dfrac{\partial \underline{\mathrm{T}}(b,y)}{\partial y} = \underline{\mathrm{T}}(b,y)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{\mathrm{R}}(y,b)] \\[2ex] -\dfrac{\partial \underline{\mathrm{H}}_\eta(b,y)}{\partial y} = \underline{\mathrm{H}}_\eta(b,y)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{\mathrm{R}}(y,b)] + [\underline{h}_e(y,+) + \underline{h}_e(y,-)\,\underline{\mathrm{R}}(y,b)] \end{cases}$$



The integration starts at level z with initial values (iia) (of par. A, above) and proceeds upward to level x using (iiia). Along the way the m×m matrices $\underline{\mathrm{R}}(y_j,b)$, $\mathrm{T}(b,y_j)$ are stored along with the 1×m vectors $\underline{H}_\eta(b,y_j)$, $j = 0,\ldots,n$.

III. Using the stored results of stages I and II above, find for $j = 0,\ldots,n$,

$$\begin{cases} \underline{T}(b,y_j,a) = \underline{\mathrm{T}}(b,y_j)[\underline{\mathrm{I}}_m - \underline{\mathrm{R}}(y_j,a)\,\underline{\mathrm{R}}(y_j,b)]^{-1} \\[1ex] \underline{R}(b,y_j,a) = \underline{T}(b,y_j,a)\,\underline{\mathrm{R}}(y_j,a) \\[1ex] \underline{H}_\eta(b,y_j,a) = [\underline{\mathrm{H}}_\eta(b,y_j) + \underline{\mathrm{H}}_\eta(a,y_j)\,\underline{\mathrm{R}}(y_j,b)][\underline{\mathrm{I}}_m - \underline{\mathrm{R}}(y_j,a)\,\underline{\mathrm{R}}(y_j,b)]^{-1} \end{cases}$$

and also

$$\begin{cases} \underline{T}(a,y_j,b) = \underline{\mathrm{T}}(a,y_j)[\underline{\mathrm{I}}_m - \underline{\mathrm{R}}(y_j,b)\,\underline{\mathrm{R}}(y_j,a)]^{-1} \\[1ex] \underline{R}(a,y_j,b) = \underline{T}(a,y_j,b)\,\underline{\mathrm{R}}(y_j,b) \\[1ex] \underline{H}_\eta(a,y_j,b) = [\underline{\mathrm{H}}_\eta(a,y_j) + \underline{\mathrm{H}}_\eta(b,y_j)\,\underline{\mathrm{R}}(y_j,a)][\underline{\mathrm{I}}_m - \underline{\mathrm{R}}(y_j,b)\,\underline{\mathrm{R}}(y_j,a)]^{-1} \end{cases}$$

These formulas are obtained from the imbed rules (17.35)-(17.40) with the substitution $y \to y_j$. The resultant $m \times m$ matrices and $1 \times m$ vectors are stored.

IV. Using (iiib,c) and the stored results of stage III, find for $j = 0, \ldots, n$,

$$\boxed{\begin{aligned} \underline{H}(y_j,+) &= \underline{H}(b,+)\ \underline{\mathcal{T}}(b,y_j,a) + \underline{H}(a,-)\ \underline{\mathcal{R}}(a,y_j,b) + \underline{\mathcal{H}}_\eta(b,y_j,a) \\ \underline{H}(y_j,-) &= \underline{H}(b,+)\ \underline{\mathcal{R}}(b,y_j,a) + \underline{H}(a,-)\ \underline{\mathcal{T}}(a,y_j,b) + \underline{\mathcal{H}}_\eta(a,y_j,b) \end{aligned}}$$

These results are stored.

V. Using (ib), (iib) and the results of Stage III, compute

$$\begin{cases} \underline{T}(b,a) = \underline{\mathcal{T}}(b,x,a)\ \underline{T}(x,a) \\ \underline{R}(a,b) = \underline{R}(a,x) + \underline{\mathcal{R}}(a,x,b)\ \underline{T}(x,a) \\ \underline{H}_\eta(b,a) = \underline{H}_\eta(x,a) + \underline{\mathcal{H}}_\eta(b,x,a)\ \underline{T}(x,a) \end{cases}$$

and also

$$\begin{cases} \underline{T}(a,b) = \underline{\mathcal{T}}(a,z,b)\ \underline{T}(z,b) \\ \underline{R}(b,a) = \underline{R}(b,z) + \underline{\mathcal{R}}(b,z,a)\ \underline{T}(z,b) \\ \underline{H}_\eta(a,b) = \underline{H}_\eta(z,b) + \underline{\mathcal{H}}_\eta(a,z,b)\ \underline{T}(z,b) \end{cases}$$

For these computations use the union rules (18.25)-(18.27) and (18.28)-(18.30). Recall that $x \equiv = y_0$ and $z = y_n$, so that, e.g., $\underline{\mathcal{T}}(b,x,a)$ is $\underline{\mathcal{T}}(b,y_0,a)$ and $\underline{\mathcal{T}}(a,z,b)$ is $\underline{\mathcal{T}}(a,y_n,b)$ as found in stage III. The matrices

$\underline{R}(a,x)$, $\underline{T}(x,a)$ and the vector $\underline{H}_\eta(x,a)$ are given in (ib), while $\underline{R}(b,z)$, $\underline{T}(z,b)$ and $\underline{H}_\eta(z,b)$ are supplied by (iib).

VI. Using the results of stage V and (iiib,c), compute

$$\boxed{\begin{aligned} \underline{H}(a,+) &= \underline{H}(b,+)\ \underline{T}(b,a) + \underline{H}(a,-)\ \underline{R}(a,b) + \underline{H}_\eta(b,a) \\ \underline{H}(b,-) &= \underline{H}(b,+)\ \underline{R}(b,a) + \underline{H}(a,-)\ \underline{T}(a,b) + \underline{H}_\eta(a,b) \end{aligned}}$$

This completes the computation of the required results $\underline{H}(y_j,\pm)$, $j = 0,\ldots,n$, $\underline{H}(a,+)$, and $\underline{H}(b,-)$.

C. *Checking the Results*

Preliminary checks are made by means of the boundary conditions (iv). For example, in (iva) we can compute $\underline{H}(a,+)$ and compare it with the $\underline{H}(a,+)$ found in Stage VI. In (iva) use the boundary data (ib) along with the incident datum (iiib), and observe that $\underline{H}(x,+)$ is $\underline{H}(y_0,+)$ found in Stage IV. These two values of $\underline{H}(a,+)$ found in distinct ways should agree. The check for $\underline{H}(b,-)$ is made by using (ivd) in a similar way and comparing the resultant $\underline{H}(b,-)$ with the $\underline{H}(b,-)$ candidate $\underline{H}(y_n,-)$ found in stage IV. Checks on $H(x,\pm)$ are made using (ivb,c) and the vectors $\underline{H}(x,\pm)$ found (as $\underline{H}(y_0,\pm)$ in stage IV): on the right sides of (ivb,c) use the step IV results and compare the boundary-condition-computed candidates for $\underline{H}(x,\pm)$ with the $\underline{H}(y_0,\pm)$ values of stage IV.

More thoroughgoing checks can be made, but they require more preliminary work. For example, one can integrate routinely not just the major trio (16.6)-(16.8) but the entire sextet (16.6)-(16.11). Their initial values are given in (ib). Store the additional matrices $\underline{T}(y_j,a)$, $\underline{R}(a,y_j)$ and the vectors

$\underline{H}_\eta(y_j,a)$, $j = 0,\ldots,n$. Similarly, compute the sextet (16.14)-(16.19) with additional initial values (ib), and store the additional quantities $\underline{T}(y_j,b)$, $\underline{R}(b,y_j)$ and $\underline{H}_\eta(y_j,b)$, $j = 0,\ldots,n$. Then, using these additional matrices and vectors, one can write down global interaction principles that act as generalized versions of the boundary conditions (iv) for $j = 0,\ldots,n$:

General Boundary conditions (v)

$$\begin{cases} \text{a)}\ \underline{H}(a,+) = \underline{H}(y_j,+)\,\underline{T}(y_j,a) + \underline{H}(a,-)\,\underline{R}(a,y_j) + \underline{H}_\eta(y_j,a) \\ \text{b)}\ \underline{H}(y_j,-) = \underline{H}(y_j,+)\,\underline{R}(y_j,a) + \underline{H}(a,-)\,\underline{T}(a,y_j) + \underline{H}_\eta(a,y_j) \\ \text{c)}\ \underline{H}(y_j,+) = \underline{H}(b,+)\,\underline{T}(b,y_j) + \underline{H}(y_j,-)\,\underline{R}(y_j,b) + \underline{H}_\eta(b,y_j) \\ \text{d)}\ \underline{H}(b,-) = \underline{H}(y_j,-)\,\underline{T}(y_j,b) + \underline{H}(b,+)\,\underline{R}(b,y_j) + \underline{H}_\eta(y_j,b) \end{cases}$$

These may be used to check the Stage IV results $H(y_j,\pm)$ for all $j = 0,\ldots,n$, as well as the two Stage VI results. The cases for $j = 0$ and $j = n$ were discussed above for boundary conditions (iv). Hence the generalized boundary conditions (v) contain the boundary conditions (iv) as special cases.

D. $\underline{R}$ $\underline{T}$ $\underline{H}_\eta$ *Example 1: Finitely Deep Medium with Air-Water Surface and Reflecting, Opaque Bottom*

The present example shows how the general data and boundary conditions of par. A may be trimmed or changed to handle a finitely deep arbitrarily stratified medium whose upper air-water boundary $X(a,x)$ is of infinitesimal thickness and with no fluorescing taking place within it. Moreover, the lower boundary $X(z,b)$ of $X(a,b)$ is a surface at a finite depth (perhaps even visible through the upper surface) and is assumed non-fluorescent and opaque, so that no light passes through it in either direction. The statements (i)-(iv) of par. A are now of the form

$$\text{(i) \underline{Boundary Data}, } X(a,x): \quad \begin{cases} \text{a) } \underline{R}(x,a) = \underline{r}(x,a),\ \underline{T}(a,x) = \underline{t}(a,x),\ \underline{H}_\eta(a,x) = \underline{0} \\ \text{b) } \underline{R}(a,x) = \underline{r}(a,x),\ \underline{T}(x,a) = \underline{t}(x,a),\ \underline{H}_\eta(x,a) = \underline{0} \end{cases}$$

(for $\underline{r}(x,a)$, and other air-water boundary transfer matrices, see (11.20))

$$\text{(ii) \underline{Boundary Data}, } X(z,b): \quad \begin{cases} \text{a) } \underline{R}(z,b) = \underline{r}(z,b),\ \underline{T}(b,z) = \underline{0}_m,\ \underline{H}_\eta(b,z) = \underline{0} \\ \text{b) } \underline{R}(b,z) = \underline{0}_m,\ \underline{T}(z,b) = \underline{0}_m,\ \underline{H}_\eta(z,b) = \underline{0} \end{cases}$$

($\underline{r}(z,b)$ is assumed known, e.g., for a matte wavelength-dependent reflectance)

$$\text{(iii)} \quad \begin{array}{ll} \text{\underline{Interior Data}, } X(x,z): & \{\text{a) } \underline{\rho}(y,\pm),\ \underline{\tau}(y,\pm),\ \underline{h}_e(y,\pm),\ x \le y \le z \\ \text{\underline{Incident Data}:} & \begin{cases} \text{b) } \underline{H}(a,-) \\ \text{c) } \underline{H}(b,+) = \underline{0}\ . \end{cases} \quad \text{(incident irradiances)} \end{array}$$

$$\text{(iv) \underline{Boundary Conditions}} \quad \begin{cases} X(a,x): & \begin{cases} \text{a) } \underline{H}(a,+) = \underline{H}(x,+)\ \underline{t}(x,a) + \underline{H}(a,-)\ \underline{r}(a,x) \\ \text{b) } \underline{H}(x,-) = \underline{H}(x,+)\ \underline{r}(x,a) + \underline{H}(a,-)\ \underline{t}(a,x) \end{cases} \\ X(z,b): & \begin{cases} \text{c) } \underline{H}(z,+) = \underline{H}(z,-)\ \underline{r}(z,b) \\ \text{d) } \underline{H}(b,-) = \underline{0} \end{cases} \end{cases}$$

Results Required:

$$\begin{cases} \text{a) } \underline{H}(y_j,\pm), \; j = 0,\ldots,n \\ \\ \text{b) } \underline{H}(a,+) \end{cases}$$

The six stages of the $\underline{R}\ \underline{T}\ \underline{H}_\eta$ procedure in par. B are now modified as follows:

I. Integrate the major Riccati trio (16.6)-(16.8) in the form

$$\begin{cases} \dfrac{\partial \underline{R}(y,a)}{\partial y} = \underline{R}(y,a)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\ \underline{R}(y,a)] \\ \\ \dfrac{\partial \underline{T}(a,y)}{\partial y} = \underline{T}(a,y)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] \\ \\ \dfrac{\partial \underline{H}_\eta(a,y)}{\partial y} = \underline{H}_\eta(a,y)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] + [\underline{h}_e(y,-) + \underline{h}_e(y,+)\ \underline{R}(y,a)] \end{cases}$$

The integration starts at level x with initial values (ia) (of this paragraph) and proceeds downward to level z using (iiia). Along the way the m×m matrices $\underline{R}(y_j,a)$, $\underline{T}(a,y_j)$ are stored with the 1×m vectors $\underline{H}_\eta(a,y_j)$, $j = 0,\ldots,n$. Here $\{y_j: \; j = 0,\ldots,n\}$ is the partition of $X(x,z)$ decided upon in (19.1).

II. Integrate (16.14) and (16.16):

$$- \frac{\partial \underline{R}(y,b)}{\partial y} = \underline{R}(y,b)\ [\underline{\tau}(y,+) + \underline{\rho}(y,+)\ \underline{R}(y,b)] + [\underline{\rho}(y,-) + \underline{\tau}(y,-)\ \underline{R}(y,b)]$$

$$(\underline{T}(b,y) = \underline{0}_m \;, \quad x \le y \le z)$$

$$- \frac{\partial \underline{H}_\eta(b,y)}{\partial y} = \underline{H}_\eta(b,y)\ [\underline{\tau}(y,+) + \underline{\rho}(y,+)\ \underline{R}(y,b)] + [\underline{h}_e(y,+) + \underline{h}_e(y,-)\ \underline{R}(y,b)]$$

The integrations starts at level z with initial values (iib) (hence it follows that $\underline{T}(b,y) \equiv \underline{0}_m$ for all y, $x \le y \le z$, as shown) and proceeds upward using (iiia). The results $\underline{R}(y_j,b)$, $\underline{H}_\eta(b,y_j)$, $j = 0,\ldots,n$ are stored.

III. Using the results of stages I, II, above, find for $j = 0,\ldots,n$,

$$\begin{cases} \underline{\mathcal{T}}(b,y_j,a) = \underline{0}_m \\ \underline{\mathcal{R}}(b,y_j,a) = \underline{0}_m \\ \underline{\mathcal{H}}_\eta(b,y_j,a) = [\underline{H}_\eta(b,y_j) + \underline{H}_\eta(a,y_j)\, \underline{R}(y_j,b)]\, [\underline{I}_m - \underline{R}(y_j,a)\, \underline{R}(y_j,b)]^{-1} \end{cases}$$

and also

$$\begin{cases} \underline{\mathcal{T}}(a,y_j,b) = \underline{T}(a,y_j)\, [\underline{I}_m - \underline{R}(y_j,b)\, \underline{R}(y_j,a)]^{-1} \\ \underline{\mathcal{R}}(a,y_j,b) = \underline{\mathcal{T}}(a,y_j,b)\, \underline{R}(y_j,b) \\ \underline{\mathcal{H}}_\eta(a,y_j,b) = [\underline{H}_\eta(a,y_j) + \underline{H}_\eta(b,y_j)\, \underline{R}(y_j,a)]\, [\underline{I}_m - \underline{R}(y_j,b)\, \underline{R}(y_j,b)]^{-1} \end{cases}$$

These formulas are obtained from the imbed rules (17.35)-(17.40) with the substitution $y \to y_j$. The results are stored. Observe that when $j = 0$, then $y_j = x$ and we set $\underline{H}_\eta(a,x) = \underline{0}$ by (ia). Moreover, when $j = n$, then $y_j = z$, and we set $\underline{H}_\eta(b,z) = \underline{0}$ by (iia). Similarly, $\underline{R}(x,a)$, $\underline{R}(z,b)$ have the values $\underline{r}(x,a)$, $\underline{r}(z,b)$ by (ia), (iia), respectively. Observe that, although $\underline{H}_\eta(b,z) = \underline{0}$ and $\underline{H}_\eta(a,x) = \underline{0}$, the integration from z upward or x downward will produce non-zero $\underline{H}_\eta(b,y_j,a)$ or $\underline{H}_\eta(a,y_j,b)$ if sources are present in X[x,z].

IV. Using the stored results of stage III and (iiib,c) find for j = 0,...,n

$$\boxed{\begin{aligned} \underline{H}(y_j,+) &= \underline{H}(a,-)\ \underline{\mathcal{R}}(a,y_j,b) + \underline{\mathcal{H}}_\eta(b,y_j,a) \\ \underline{H}(y_j,-) &= \underline{H}(a,-)\ \underline{\mathcal{T}}(a,y_j,b) + \underline{\mathcal{H}}_\eta(a,y_j,b) \end{aligned}}$$

These results are stored.

V. Using (ib), (iib) and the results of stage III, compute

$$\begin{cases} \underline{T}(b,a) = \underline{0}_m \\ \underline{R}(a,b) = \underline{r}(a,x) + \underline{\mathcal{R}}(a,x,b)\ \underline{t}(x,a) \\ \underline{H}_\eta(b,a) = \underline{\mathcal{H}}_\eta(b,x,a)\ \underline{t}(x,a) \end{cases}$$

and also

$$\begin{cases} \underline{T}(a,b) = \underline{0}_m \\ \underline{R}(b,a) = \underline{0}_m \\ \underline{H}_\eta(a,b) = \underline{0} \end{cases}$$

VI. Using the results of stage V and (iiib,c), compute

$$\boxed{\underline{H}(a,+) = \underline{H}(a,-)\ \underline{R}(a,b) + \underline{H}_\eta(b,a)}$$

This completes the computation of the required results $\underline{H}(y_j,\pm)$, j = 0,...,n, and $\underline{H}(a,+)$. The checks are carried out as in par. C, above. The new general boundary conditions (v) of par. C that are the basis of the general check now take the following forms for depths y_j, j = 0,...,n

General boundary conditions (v)

$$\begin{cases} \text{a)} & \underline{H}(a,+) = \underline{H}(y_j,+)\,\underline{T}(y_j,a) + \underline{H}(a,-)\,\underline{R}(a,y_j) + \underline{H}_\eta(y_j,a) \\ \text{b)} & \underline{H}(y_j,-) = \underline{H}(y_j,+)\,\underline{R}(y_j,a) + \underline{H}(a,-)\,\underline{T}(a,y_j) + \underline{H}_\eta(a,y_j) \\ \text{c)} & \underline{H}(y_j,+) = \underline{H}(y_j,-)\,\underline{R}(y_j,b) + \underline{H}_\eta(b,y_j) \end{cases}$$

As can be seen by comparing this with its earlier version in par. C, conditions a), b) are unchanged and condition c) has been modified to the present form (since $\underline{H}(b,+) = \underline{0}$). Condition d) has been omitted since all terms in it are zero vectors or zero matrices by virtue of the data and boundary conditions. We now check this in detail: Observe by union rule (18.28) that $T(y_j,b) = \underline{\mathcal{T}}(y_j,z,b)\,\underline{T}(z,b)$, and recall that $\underline{T}(z,b) = \underline{0}_m$ by (iib). This accounts for the missing transmittance term in d). By (iiic), $\underline{H}(b,+) = \underline{0}$, which accounts for the missing reflectance term in d). Finally, by the union rule (18.30), $\underline{H}_\eta(y_j,b) = \underline{H}_\eta(z,b) + \underline{\mathcal{H}}_\eta(y_j,z,b)\,\underline{T}(z,b)$. By (iib), $\underline{H}_\eta(z,b) = \underline{0}$ and as noted already above, $\underline{T}(z,b) = \underline{0}_m$. Thus term d) in (v) above is omitted from present form of the check.

E. $\underline{\mathcal{R}}\ \underline{\mathcal{T}}\ \underline{\mathcal{H}}_\eta$ *Example 2: Finitely Deep Stratified Natural Hydrosol Resting on an Infinitely Deep Homogeneous Lower Hydrosol*

We now consider a natural hydrosol of the form $X(a,\infty) = X(a,x) \cup X(x,z) \cup X(z,\infty)$ where $X(z,\infty)$ is an infinitely deep homogeneous hydrosol.

The main point of the present example is to show how to handle the computational problem presented by the infinitely deep homogeneous layer $X(z,\infty)$ that starts at some finite depth z below the surface. We certainly cannot integrate over such a medium; but we have the results of our eigenmatrix analysis in section 13 to fall back upon. Between the surface X(a,x) and level z the medium X(x,z) can fluoresce and have optical properties that vary generally with depth. The interface at level z between X(x,z) and X(z,∞) is by choice here nonreflecting. The data and boundary value statements (i)-(iv) of par. B reduce to the present case, as follows:

(i) Boundary data, X(a,x):
$$\begin{cases} \text{a) } R(x,a) = r(x,a),\ \underline{T}(a,x) = \underline{t}(a,x),\ \underline{H}_\eta(a,x) = \underline{0} \\ \text{b) } \underline{R}(a,x) = \underline{r}(a,x),\ \underline{T}(x,a) = \underline{t}(x,a),\ \underline{H}_\eta(x,a) = \underline{0} \end{cases}$$

(ii) Boundary Data, X(z,∞):
$$\begin{cases} \text{a) } R(z,\infty) = \underline{R}_-(\infty),\ \underline{T}(\infty,z) = \underline{0}_m,\ \underline{H}_\eta(\infty,z) = \underline{0} \\ \text{b) } \underline{R}(\infty,z) = \underline{0}_m,\ \underline{T}(z,\infty) = \underline{0}_m,\ \underline{H}_\eta(z,\infty) = \underline{0} \end{cases}$$

The form of $\underline{R}_-$ will be given below. The air-water surface transfer matrices $\underline{r}(x,a)$, $\underline{r}(a,x)$, $\underline{t}(x,a)$, $\underline{t}(a,x)$ are given.

(iii)

Interior Data X(x,z): $\left\{ \text{a) } \rho(y,\pm),\ \tau(y,\pm),\ \underline{h}_e(y,\pm),\ x \le y \le z \right.$

Incident Data:
$$\begin{cases} \text{b) } \underline{H}(a,-) \\ \text{c) } \underline{H}(\infty,+) = \underline{0} \end{cases}$$

(iv) Boundary Conditions

$$X(a,x): \begin{cases} \text{a) } \underline{H}(a,+) = \underline{H}(x,+)\ \underline{t}(x,a) + \underline{H}(a,-)\ \underline{r}(a,x) \\ \text{b) } \underline{H}(x,-) = \underline{H}(x,+)\ \underline{r}(x,a) + \underline{H}(a,-)\ \underline{t}(a,x) \end{cases}$$

$$X(z,\infty): \begin{cases} \text{c) } \underline{H}(z,+) = \underline{H}(z,-)\ \underline{R}(z,\infty) \\ \text{d) } \underline{H}(\infty,-) = \underline{0} \end{cases}$$

The matrix $\underline{R}(z,\infty)$ is that given by $\underline{R}_-(\infty)$ in (iia), and specified, below.

Results Required:
$$\begin{cases} \text{a) } \underline{H}(y_j,\pm)\ ,\ j = 0,\ldots,n \\ \text{b) } \underline{H}(a,+) \end{cases}$$

The six stages of the $\underline{R}\ \underline{T}\ \underline{H}_\eta$ procedure in par. B now take the forms:

I. Integrate the major Riccati trio (16.6)-(16.8):

$$\begin{cases} \dfrac{\partial \underline{R}(y,a)}{\partial y} = \underline{R}(y,a)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\ R(y,a)] \\ \dfrac{\partial \underline{T}(a,y)}{\partial y} = \underline{T}(a,y)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] \\ \dfrac{\partial \underline{H}_\eta(a,y)}{\partial y} = \underline{H}_\eta(a,y)\ [\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] + [\underline{h}_e(y,-) + \underline{h}_e(y,+)\ \underline{R}(y,a)] \end{cases}$$

Therefore this first stage is the same as in Example 1.

The integration starts at level x with initial values (ia) of the present paragraph and proceeds downward to level z using (iiia). Along the way the m×m matrices $\underline{R}(y_j,a)$, $T(a,y_j)$ are stored with the 1×m vectors $\underline{H}_\eta(a,y_j)$, $j = 0,\ldots,n$. Here, $\{y_j\colon\ j = 0,\ldots,n\}$ is the partition of $X(x,z)$ decided upon in (19.1)

II. Integrate (16.14) and (16.16)

$$\begin{cases} -\dfrac{\partial \underline{R}(y,\infty)}{\partial y} = \underline{R}(y,\infty)\,[\underline{\tau}(y,+) + \rho(y,+)\,\underline{R}(y,\infty)] + [\rho(y,-) + \underline{\tau}(y,-)\,\underline{R}(y,\infty)] \\ \underline{T}(\infty,y) = \underline{0}_m \;,\; x \le y \le z \\ -\dfrac{\partial \underline{H}_\eta(\infty,y)}{\partial y} = \underline{H}_\eta(\infty,y)\,[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{R}(y,\infty)] + [\underline{h}_e(y,+) + \underline{h}_e(y,-)\,\underline{R}(y,\infty)] \end{cases}$$

The integration starts at level z with initial values (iia) [hence it follows that $\underline{T}(\infty,y) = \underline{0}_m$ for all y, $x \le y \le z$, as shown] and proceeds upward using (iiia). The matrices $\underline{R}(y_j,\infty)$ and vectors $\underline{H}_\eta(\infty,y_j)$, $j = 0,\ldots,n$ are stored. We generally expect $\underline{H}_\eta(\infty,y)$ to be nonzero owing to the presence of true sources $\underline{h}_e(y,\pm) \neq \underline{0}$ for y in the range $x \le y \le z$, and despite the initial condition (iia), $\underline{H}_\eta(\infty,z) = \underline{0}$.

III. Using the results of steps I, II above, find the $j = 0,\ldots,n$

$$\begin{cases} \underline{\mathcal{T}}(\infty,y_j,a) = \underline{0}_m \\ \underline{\mathcal{R}}(\infty,y_j,a) = \underline{0}_m \\ \underline{\mathcal{H}}_\eta(\infty,y_j,a) = [\underline{H}_\eta(\infty,y_j) + \underline{H}_\eta(a,y_j)\underline{R}(y_j,\infty)]\,[\underline{I}_m - \underline{R}(y_j,a)\,\underline{R}(y_j,\infty)]^{-1} \end{cases}$$

and also

$$\begin{cases} \underline{\mathcal{T}}(a,y_j,\infty) = \underline{T}(a,y_j)\,[\underline{I}_m - \underline{R}(y_j,\infty)\,\underline{R}(y_j,a)]^{-1} \\ \underline{\mathcal{R}}(a,y_j,\infty) = \underline{\mathcal{T}}(a,y_j,\infty)\,\underline{R}(y_j,\infty) \\ \underline{\mathcal{H}}_\eta(a,y_j,\infty) = [\underline{H}_\eta(a,y_j) + \underline{H}_\eta(\infty,y_j)\,\underline{R}(y_j,a)]\,[\underline{I}_m - \underline{R}(y_j,\infty)\,\underline{R}(y_j,a)]^{-1} \end{cases}$$

These formulas are obtained from the imbed rules (17.35)-(17.40) with the substitutions $y \to y_j$ and $b \to \infty$. The results are stored. When $j = 0$, then $y_j = x$ and we set $\underline{H}_\eta(a,x) = \underline{0}$ by (ia) (no internal true sources in the air-water surface). Moreover, when $j = n$, then $y_j = z$ and we set $\underline{H}_\eta(\infty,z) = \underline{0}$ by (iia) (no true sources below depth z). Similarly, $\underline{R}(x,a)$ is set equal to $\underline{r}(x,a)$ and $\underline{R}(z,\infty)$ is set equal to $\underline{R}_-(\infty)$, by (ia) and (iia), respectively.

IV. Using the stored results of stage III and (iiib,c), find for $j = 0,\ldots,n$,

$$\boxed{\begin{aligned} \underline{H}(y_j,+) &= \underline{H}(a,-)\ \underline{\mathcal{R}}(a,y_j,\infty) + \underline{\mathcal{H}}_\eta(\infty,y_j,a) \\ \underline{H}(y_j,-) &= \underline{H}(a,-)\ \underline{\mathcal{T}}(a,y_j,\infty) + \underline{\mathcal{H}}_\eta(a,y_j,\infty) \end{aligned}}$$

These results are stored. Observe the absence of any upward incident irradiances on $X(a,\infty)$, as may be expected.

V. Using (ib), (iib) and the results of Stage III, compute

$$\left\{\begin{aligned} \underline{T}(\infty,a) &= \underline{0}_m \\ \underline{R}(a,\infty) &= \underline{r}(a,x) + \underline{\mathcal{R}}(a,x,\infty)\ \underline{t}(x,a) \\ \underline{H}_\eta(\infty,a) &= \underline{\mathcal{H}}_\eta(\infty,x,a)\ \underline{t}(x,a) \end{aligned}\right.$$

and also

$$\begin{cases} \underline{T}(a,\infty) = \underline{0}_m \\ \underline{R}(\infty,a) = \underline{0}_m \\ \underline{H}_\eta(a,\infty) = \underline{0}_m \end{cases}$$

VI. Using the results of Stage V and (iiib,c), compute

$$\boxed{\underline{H}(a,+) = \underline{H}(a,-)\ \underline{R}(a,\infty) + \underline{H}_\eta(\infty,a)}$$

The new general boundary condition check (v) of par. C takes the following form for depths y_j, $j = 0,\ldots,n$:

a) $\underline{H}(a,+) = \underline{H}(y_j,+)\ \underline{T}(y_j,a) + \underline{H}(a,-)\ \underline{R}(a,y_j) + \underline{H}_\eta(y_j,a)$

b) $\underline{H}(y_j,-) = \underline{H}(y_j,+)\ \underline{R}(y_j,a) + \underline{H}(a,-)\ \underline{T}(a,y_j) + \underline{H}_\eta(a,y_j)$

c) $\underline{H}(y_j,+) = \underline{H}(y_j,-)\ \underline{R}(y_j,\infty) + \underline{H}_\eta(\infty,y_j)$

On comparing the present six stages with those in Example 1, the various changes entailed in going from a finite to an infinite medium are now precisely stated. It is seen that the salient change is in $\underline{R}(z,b)$ which is now the reflectance $\underline{R}(z,\infty)$ of an infinitely deep homogeneous slab $X(z,\infty)$. By (15.57) and (15.69) we have

$$R(z,\infty) \equiv \underline{R}_-(\infty) = -\underline{\underline{E}}_{-+}\ \underline{\underline{E}}_{++}^{-1} = \underline{\underline{F}}_{--}^{-1}\ \underline{\underline{F}}_{-+}$$

$$= [\underline{\underline{F}}_{--}^{-1}\ \underline{\underline{k}}_-\ \underline{\underline{F}}_{--} - \underline{\tau}(-)]\ \underline{\rho}^{-1}(+)$$

where $\underline{\rho}(+)$, $\underline{\tau}(-)$ are the depth-independent m×m local transfer matrices for the homogeneous layer $X(z,\infty)$. Moreover, $\underline{F}_{--}$ and $\underline{F}_{-+}$ are m×m submatrices of $\underline{F}$ defined in (13.20). The representation $-\underline{E}_{-+}\,\underline{E}_{++}^{-1}$ of $\underline{R}_{-}(\infty)$ is perhaps easiest to use.

It is possible to relax the homogeneous and source-free condition on $X(z,\infty)$, and still obtain in a workable way the reflectance $\underline{R}(z,\infty)$, and source-generated vector $\underline{H}_{\eta}(\infty,z)$. Suppose the interior data in $X(z,\infty)$ are arbitrary but specified non-constant functions of y: $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$, $\underline{h}_{e}(y,\pm)$, $z \le y < \infty$. Now, for all natural hydrosols the entries of $\underline{\rho}(y,\pm)$ and $\underline{\tau}(y,\pm)$ are non-zero for all y, and in fact are bounded away from zero, so that they have positive minima. Hence we would expect the effectively infinitely deep layer $X(z,\infty)$ of a lake or sea to have the property that $\underline{T}(z,u)$, where $z < u$, becomes small as u increases, i.e., that $\underline{T}(z,u) \to \underline{0}_{m}$ as $u \to \infty$, and, moreover, that $\underline{R}(z,u)$ approaches a finite matrix $\underline{R}(z,\infty)$ as $u \to \infty$. We have a way of finding this limit. We simply program the downward sweep Riccati sextet (16.6)-(16.11) (with 'x' replaced by 'z', and 'y' by 'u', so that $z < u$). Then we would expect to have well-behaved (bounded, non-oscillatory) solutions $\underline{R}(z,u)$ and $\underline{H}_{\eta}(u,z)$ (as given by (16.10), (16.11)), and we also expect that they would go to finite limits fairly quickly as $u \to \infty$. Although $X(z,\infty)$ is optically infinitely deep and the depth variations of $\underline{\rho}_{\pm}$, $\underline{\tau}_{\pm}$ and $\underline{h}_{e\pm}$ can be quite wild over limited depth intervals, the integrations of (16.10)-(16.11) would converge reasonably quickly for all practical purposes. For example convergence to within three or four significant figures of their limits would occur at depths u about 10 optical depths below z, i.e. when $\alpha_{m}(u-z) \simeq 10$, where α_{m} is the minimum of $\alpha(y)$ over the range $x \le y \le z$.

F. $\underline{R}\ \underline{T}\ \underline{H}_\eta$ *Example 3: Infinitely Deep Homogeneous Natural Hydrosol with Uniformly Distributed Internal Sources*

The present medium $X(a,\infty)$ is a special case of that considered in Example 2. Once again we can write $X(a,\infty) = X(a,x) \cup X(x,z) \cup X(z,\infty)$, but now $X(x,\infty) = X(x,z) \cup X(z,\infty)$ is homogeneous, from just under the air-water surface on down to ∞. The lower boundary $X(z,\infty)$ is homogeneous and source-free as in Example 2; but now we imagine z to be arbitrarily large, so that the boundary data of (iib), Example 2 hold for all $z > x$. This feature of $X(x,\infty)$ allows us to retain the framework of Example 2 for each finite z. It allows a sequence of Example 2 problems to approach the present problem. Ultimately this produces a simplification of various stages in the $\underline{R}\ \underline{T}\ \underline{H}_\eta$ solution procedure we have been studying, and these simplifications will be brought out in the stage statements below. In particular it will be possible in principle to replace all Riccati sweeps by closed algebraic forms found through the eigenmatrix formalism. Except for the distinguishing feature about the present $X(z,\infty)$ defined above (z arbitrarily large) the data and boundary value statements are essentially the same as those of Example 2. To verify this, and for future reference, these are fully written out here:

(i) <u>Boundary Data</u> $X(a,x)$:

$$\begin{cases} \text{a) } \underline{R}(x,a) = \underline{r}(x,a),\ \underline{T}(a,x) = \underline{t}(a,x),\ \underline{H}_\eta(a,x) = \underline{0} \\ \text{b) } \underline{R}(a,x) = \underline{r}(a,x),\ \underline{T}(x,a) = \underline{t}(x,a),\ \underline{H}_\eta(x,a) = \underline{0} \end{cases}$$

(ii) <u>Boundary Data</u> $X(z,\infty)$: for all z, $z > x$

$$\begin{cases} \text{a) } \underline{R}(z,\infty) = \underline{R}_-(\infty),\ \underline{T}(\infty,z) = \underline{0}_m,\ \underline{H}_\eta(\infty,z) = \underline{0} \\ \text{b) } \underline{R}(\infty,z) = \underline{0}_m,\ \underline{T}(z,\infty) = \underline{0}_m,\ \underline{H}_\eta(z,\infty) = \underline{0} \end{cases}$$

(The form of $\underline{R}_-(\infty)$ is $-\underline{E}_{-+}\,\underline{E}_{++}^{-1}$, as given below stage VI of Example 2; cf. (15.57), (15.69))

(iii)

Interior Data $X(x,z)$: $\left\{ \text{a) } \underline{\rho}(\pm),\ \underline{\tau}(\pm),\ \underline{h}_e(\pm) \right.$

(For all z, $z > x$, i.e., $\underline{\rho}(y,\pm)$, $\underline{\tau}(y,\pm)$, $\underline{h}_e(y,\pm)$ are constant on $X(x,z)$, for all choices of z.)

Incident Data:
$$\begin{cases} \text{a) } \underline{H}(a,-) \\ \text{b) } \underline{H}(\infty,+) = \underline{0} \end{cases}$$

(iv) Boundary Condition:
$$\begin{cases} X(a,x): & \begin{cases} \text{a) } \underline{H}(a,+) = \underline{H}(x,+)\ \underline{t}(x,a) + \underline{H}(a,-)\ \underline{r}(a,x) \\ \text{b) } \underline{H}(x,-) = \underline{H}(x,+)\ \underline{r}(x,a) + \underline{H}(a,-)\ \underline{t}(a,x) \end{cases} \\ X(z,\infty) & \begin{cases} \text{c) } \underline{H}(z,+) = \underline{H}(z,-)\ \underline{R}(z,\infty), \text{ for all } z,\ z > x \\ \text{d) } \underline{H}(\infty,-) = \underline{0} \end{cases} \end{cases}$$

The six stages of the $\underline{R}\ \underline{T}\ \underline{H}_\eta$ procedure on par. E now take the forms

I. Integrate the major Riccati trio (16.6)-(16.8):

$$\begin{cases} \dfrac{\partial \underline{R}(y,a)}{\partial y} = \underline{R}(y,a)\ [\underline{\tau}(-) + \underline{\rho}(-)\ \underline{R}(y,a)] + [\underline{\rho}(+) + \underline{\tau}(+)\ \underline{R}(y,a)] \\ \dfrac{\partial \underline{T}(a,y)}{\partial y} = \underline{T}(a,y)\ [\underline{\tau}(-) + \underline{\rho}(-)\ \underline{R}(y,a)] \\ \dfrac{\partial \underline{H}_\eta(a,y)}{\partial y} = \underline{H}_\eta(a,y)\ [\underline{\tau}(-) + \underline{\rho}(-)\ \underline{R}(y,a)] + [\underline{h}_e(-) + \underline{h}_e(+)\ \underline{R}(y,a)] \end{cases}$$

Integrate the trio from x (= y_0) down to any finite z (= y_n). Use initial values in (ia). Record $\underline{R}(y_j,a)$, $\underline{T}(a,y_j)$ and $\underline{H}_\eta(a,y_j)$, j = 0,...,n.

Observe that the local properties $\underline{\rho}$, $\underline{\tau}$, $\underline{h}_e$ are independent of depth

II. $$\begin{cases} \underline{R}(y,\infty) = \underline{R}_-(\infty) \quad , \quad x \le y < \infty \\ \underline{T}(\infty,y) = \underline{0}_m \quad , \quad x \le y < \infty \\ \underline{H}_\eta(\infty,y) = -[\underline{h}_e(+) + \underline{h}_e(-)\ \underline{R}_-(\infty)]\ [\underline{\tau}(+) + \underline{\rho}(+)\ \underline{R}_-(\infty)]^{-1}\ (\equiv \underline{H}_\eta(\infty,+)) \end{cases}$$

The particular value $\underline{R}_-(\infty)$ of $\underline{R}(y,\infty)$ is $-\underline{E}_{-+}\ \underline{E}_{++}^{-1}$ and follows from the discussion of Example 2 in par. E. The reason that $\underline{T}(\infty,y)$ is $\underline{0}_m$ was covered also in Example 2. The form for $\underline{H}_\eta(\infty,y)$ follows rigorously from setting to $\underline{0}$ the derivative $\partial\underline{H}_\eta(\infty,y)/\partial y$ in Stage II of Example 2 and solving for $\underline{H}_\eta(\infty,y)$. This is based on the observation that $\underline{H}_\eta(z,y)$, for very large fixed depth z, is nearly independent of y because the build-up of $\underline{H}_\eta(z,y)$ through $\underline{h}_e(\pm)$ over the depth interval from z to y has then nearly completely taken place, even though the boundary datum (iia), namely $\underline{H}_\eta(\infty,z) = \underline{0}$, began the integration of the differential equation for $\underline{H}_\eta(z,y)$. Store this fixed value of $\underline{H}_\eta(\infty,y)$ and henceforth denote it by '$\underline{H}_\eta(\infty,+)$'. Note how boundary data (iia) and the interior data (iiia) are used throughout stage II to establish the above three statements.

III. Using the results of stages I, II above, find for j = 0,...,n

$$\begin{cases} \underline{\mathcal{T}}(\infty,y_j,a) = \underline{0}_m \\ \underline{\mathcal{R}}(\infty,y_j,a) = \underline{0}_m \\ \underline{\mathcal{H}}_\eta(\infty,y_j,a) = [\underline{H}_\eta(\infty,+) + \underline{H}_\eta(a,y_j)\ \underline{R}_-(\infty)]\ [\underline{I}_m - \underline{R}(y_j,a)\ \underline{R}_-(\infty)]^{-1} \end{cases}$$

and also

$$\begin{cases} \underline{\mathcal{T}}(a,y_j,\infty) = \underline{T}(a,y_j)\ [\underline{I}_m - \underline{R}_-(\infty)\ \underline{R}(y_j,a)]^{-1} \\ \underline{\mathcal{R}}(a,y_j,\infty) = \underline{\mathcal{T}}(a,y_j,\infty)\ \underline{R}_-(\infty) \\ \underline{\mathcal{H}}_\eta(a,y_j,\infty) = [\underline{H}_\eta(a,y_j) + \underline{H}_\eta(\infty,+)\ \underline{R}(y_j,a)]\ [\underline{I}_m - \underline{R}_-(\infty)\ R(y_j,a)]^{-1} \end{cases}$$

These formulas are obtained from the imbed rules (17.35)-(17.40) with the substitutions $y \to y_j$ and $b \to \infty$. The results are stored. Observe that $\underline{H}_\eta(a,x) = \underline{0}$ by (ia) and $\underline{H}_\eta(\infty,z) = \underline{0}$ by (iia), for every z, z > x (in the sequence of approximations of X(x,z) to X(x,∞)). Moreover, $\underline{R}(x,a) = \underline{r}(x,a)$ by (ia) and $\underline{R}(z,\infty)$ has been replaced by $\underline{R}_-(\infty)$ by virtue of (iia).

IV. Using the stored results of Stage III and (iiib,c) find for j = 0,...,n

$$\boxed{\begin{aligned} \underline{H}(y_j,+) &= \underline{H}(a,-)\ \underline{\mathcal{R}}(a,y_j,\infty) + \underline{\mathcal{H}}_\eta(\infty,y_j,a) \\ \underline{H}(y_j,-) &= \underline{H}(a,-)\ \underline{\mathcal{T}}(a,y_j,\infty) + \underline{\mathcal{H}}_\eta(a,y_j,\infty) \end{aligned}}$$

These results are stored.

V. Using (ib), (iib) and the results of Stage III, compute

$$\begin{cases} \underline{T}(\infty,a) = \underline{0}_m \\ \underline{R}(a,\infty) = \underline{r}(a,x) + \underline{\mathcal{R}}(a,x,\infty)\ \underline{t}(x,a) \\ \underline{H}_\eta(\infty,a) = \underline{\mathcal{H}}_\eta(\infty,x,a)\ \underline{t}(x,a) \end{cases}$$

and also

$$\begin{cases} \underline{T}(a,\infty) = \underline{0}_m \\ \underline{R}(\infty,a) = \underline{0}_m \\ \underline{H}_\eta(a,\infty) = \underline{0}_m \end{cases}$$

VI. Using the results of Stage V and (iiia), compute

$$\boxed{\underline{H}(a,+) = \underline{H}(a,-)\ \underline{R}(a,\infty) + \underline{H}_\eta(\infty,a)}$$

The boundary value checks are as given below Stage VI, Example 2.

It is of interest to examine some special values of $\underline{H}(y_j,\pm)$, as given by Stage IV. Our intent is to establish the assertions, stated at the outset, that the present problem may be solved without resort to numerical integrations of differential integrations. Consider the case of $y_0 = x$, so that $\underline{H}(x,\pm)$ are the irradiances just under the air-water surface. We have from IV:

$$\boxed{\begin{aligned} \underline{H}(x,+) &= \underline{H}(a,-)\ \underline{\mathcal{R}}(a,x,\infty) + \underline{\mathcal{H}}_\eta(\infty,x,a) \\ \underline{H}(x,-) &= \underline{H}(a,-)\ \underline{\mathcal{T}}(a,x,\infty) + \underline{\mathcal{H}}_\eta(a,x,\infty) \end{aligned}} \qquad (19.2)$$

Here, by Stage III,

$$\underline{T}(a,x,\infty) = \underline{t}(a,x)\,[\underline{I}_m - \underline{R}_-(\infty)\,\underline{r}(x,a)]^{-1}$$

$$\underline{R}(a,x,\infty) = \underline{T}(a,x,\infty)\,\underline{R}_-(\infty) \quad , \quad \underline{R}_-(\infty) = -\underline{E}_{-+}\,\underline{E}_{++}^{-1}$$

Moreover,

$$\underline{H}_\eta(\infty,x,a) = \underline{H}_\eta(\infty,+)\,[\underline{I}_m - \underline{r}(x,a)\,\underline{R}_-(\infty)]^{-1}$$

$$\underline{H}_\eta(a,x,\infty) = \underline{H}_\eta(\infty,x,a)\,\underline{r}(x,a)$$

where, by Stage II,

$$\underline{H}_\eta(\infty,+) = -[\underline{h}_e(+) + \underline{h}_e(-)\,\underline{R}_-(\infty)]\,[\underline{\tau}(+)\,\underline{\rho}(+)\,\underline{R}_-(\infty)]^{-1}$$

This explicit array of formulas shows that $\underline{H}(x,\pm)$ in (19.2) can be obtained without any integrations: all that need be known are $\underline{R}_-(\infty)$ ($= -\underline{E}_{-+}\,\underline{E}_{++}^{-1}$) and $\underline{t}(a,x)$, $\underline{r}(x,a)$ of the air-water surface. $\underline{H}(a,+)$ in Stage VI is also found without integration, since $\underline{R}(a,b)$ in Stage VI is obtained via $\underline{R}_-(\infty)$, the air-water surface's transfer functions, and $\underline{H}_\eta(\infty,x,a)$. Hence when working with a homogeneous medium the main concepts needed for computations are the eigenstructures $\underline{E}$ and $\underline{k}$ of Section 13.

The preceding observation may be illustrated further. Let us find, without the aid of a differential equation integration, the irradiance field $\underline{H}(y,\pm)$ at an arbitrary depth y, $x < y < \infty$ in $X(x,\infty)$. From (17.3), (17.4), for the case of $z = \infty$ and incident datum (iiib) we have

$$\boxed{\begin{aligned} H(y,+) &= \underline{H}(x,-)\, \mathcal{R}(x,y,\infty) + \underline{\mathcal{H}}_\eta(\infty,y,x) \\ \underline{H}(y,-) &= \underline{H}(x,-)\, \underline{\mathcal{T}}(x,y,\infty) + \underline{\mathcal{H}}_\eta(x,y,\infty) \end{aligned}} \tag{19.3}$$

where by (17.7) and (17.8),

$$\underline{\mathcal{T}}(x,y,\infty) = \underline{T}(x,y)\, [\underline{I}_m - \underline{R}_-(\infty)\, \underline{R}(y,x)]^{-1}$$

$$\underline{\mathcal{R}}(x,y,\infty) = \underline{\mathcal{T}}(x,y,\infty)\, \underline{R}_-(\infty)$$

and by (17.9), (17.10)

$$\begin{aligned} \underline{\mathcal{H}}_\eta(\infty,y,x) &= [\underline{H}_\eta(\infty,+) + \underline{H}_\eta(x,y)\, \underline{R}_-(\infty)]\, [\underline{I}_m - \underline{R}(y,x)\, \underline{R}_-(\infty)]^{-1} \\ \underline{\mathcal{H}}_\eta(x,y,\infty) &= [\underline{H}_\eta(x,y) + \underline{H}_\eta(\infty,+)\, \underline{R}(x,y)]\, [\underline{I}_m - \underline{R}_-(\infty)\, \underline{R}(y,x)]^{-1} \end{aligned} \tag{19.4}$$

Now, $\underline{T}(x,y)$ and $\underline{R}(y,x)$ occurring in these formulas, particularly for $\underline{\mathcal{T}}(x,y,\infty)$ and $\underline{\mathcal{R}}(x,y,\infty)$, can be obtained from the eigenmatrix theory. See (15.56b,c) and (17.91). Thus $\underline{\mathcal{R}}(x,y,\infty)$ and $\underline{\mathcal{T}}(x,y,\infty)$ can be determined without numerical integration, and for arbitrary depths y (not just those y_j, $j = 0,\ldots,n$ of Example 2). The representations of $\underline{\mathcal{R}}$ and $\underline{\mathcal{T}}$ given specifically in terms of the $\underline{\mathcal{M}}$-matrix can also be used (cf. (17.96)).

As for the $\underline{H}_\eta(x,y)$ and $\underline{H}_\eta(\infty,+)$ terms in (19.4), above, if sources are present then we require $\underline{H}_\eta(x,y)$ to complete the calculation of $H(y,\pm)$. It is possible in the present homogeneous setting to evaluate $H_\eta(x,y)$ in closed form using the calculus. From (15.52b) we have

$$\underline{H}_\eta(x,y) = -\underline{H}_e(y,x,-)\, \underline{\underline{\mathcal{M}}}^{-1}(y,x) \tag{19.5}$$

where $\underline{H}_e(y,x,-)$ is the downward (-) part of $\underline{H}_e(y,x) = [\underline{H}_e(y,x,+), \underline{H}_e(y,x,-)]$, as originally encountered in (14.17a). Hence we require the evaluation of $\underline{H}_e(y,x,-)$.

Now, from (14.16), for arbitrarily located x,y (i.e., we can have $x < y$ or $y < x$, as long as x,y are in the region of definition of $\underline{h}_e(s)$, $\underline{\rho}(s,\pm)$, and $\underline{\tau}(s,\pm)$):

$$\underline{H}_e(x,y) = \int_x^y \underline{h}_e(s)\, \underline{\mathcal{M}}(s,y)\, ds \tag{19.6}$$

For $\underline{H}_e(y,x)$ we would simply interchange 'x' and 'y' in the results below. Using the eigenmatrix representation (13.33) of $\underline{\mathcal{M}}(s,y)$ and the postulated s-independence of $\underline{h}_e(s)$, (19.6) becomes

$$\begin{aligned}
\underline{H}_e(x,y) &= \underline{h}_e \int_x^y \underline{\mathcal{E}} \exp\{\underline{k}(y-s)\}\underline{\mathcal{F}}\, ds \\
&= \underline{h}_e\, \underline{\mathcal{E}}[\int_x^y \exp\{\underline{k}(y-s)\}ds]\, \underline{\mathcal{F}} \\
&= -\underline{h}_e\, \underline{\mathcal{E}}\, \underline{k}^{-1}[\underline{I}_m - \exp\{\underline{k}(y-x)\}]\underline{\mathcal{F}} \\
&= \underline{h}_e\, \underline{\mathcal{E}}\, \underline{k}^{-1}\, \underline{\mathcal{F}}[\underline{\mathcal{M}}(x,y) - \underline{I}_{2m}] &\text{(19.7a)} \\
&= \underline{h}_e\, \underline{\mathcal{K}}^{-1}[\underline{\mathcal{M}}(x,y) - \underline{I}_{2m}] &\text{(19.7b)}
\end{aligned}$$

where we have used (13.14). As a check, differentiating (19.7b) with respect to y, we obtain agreement with (14.16a). Next, recall (cf. (14.17b)) that $\underline{H}_e(x,y,\pm) = [H_e(x,y,\pm,1),\ldots,H_e(x,y,\pm,m)]$. Then on the component level (19.7a) becomes,* for $j = 1,\ldots,m$,

* Alternatively, one can work out the inverse $\underline{\mathcal{K}}^{-1}$ in terms of the matrices $\underline{\rho}(\pm)$, $\underline{\tau}(\pm)$; see (13.2) and (15.14). However, since the $\underline{\mathcal{E}}$ and $\underline{\mathcal{F}}$ have presumably been evaluated, we remain with them.

$$H_e(x,y,\pm,j) =$$

$$= \sum_{i=1}^{m} \sum_{\ell=1}^{m} \{[h_e(+,i)e_{++}(i,\ell) + h_e(-,i)e_{-+}(i,\ell)]k_+^{-1}(\ell)[\exp\{k_+(\ell)(y-x)\}-1]f_{+\pm}(\ell,j)$$

$$+ [h_e(+,i)e_{+-}(i,\ell) + h_e(-,i)e_{--}(i,\ell)]k_-^{-1}(\ell)[\exp\{k_-(\ell)(y-x)\}-1]f_{-\pm}(\ell,j)\} \tag{19.8}$$

This leads to the required closed-form (i.e., fully integrated) representation of $H_e(y,x,\pm,j)$, $j = 1,..,m$, needed in (19.5) (after interchanging 'x' and 'y' in (19.8)). The reader may perhaps now agree that (19.8), except for certain theoretical analyses (for example, what is the expansion of $H_e(x,y,\pm,j)$ to first order in $(y-x)$?), may well be by-passed in practice. It would, in the case of sources present, be far more expedient simply to integrate numerically the Riccati trio in Stage I (with the substitution $a \to x$, and bare-medium initial conditions).

20. TRANSPORT SOLUTION OF THE LIGHT FIELD: $\underline{T}\ \underline{H}_\eta$ ANALYTIC PROCEDURE

A. *Introductory Comments*

We next consider the second of three transport solution procedures of the light field, namely the $\underline{T}\ \underline{H}_\eta$ analytic procedure. This procedure differs from the $\underline{R}\ \underline{T}\ \underline{H}_\eta$ procedure of section 19 in that it is analytic rather than algebraic. It uses integration sweeps of the equations governing the two (upward and downward) complete transmittances $\underline{T}$ and the two complete irradiances $\underline{H}_\eta$ to obtain these objects at various depths in the medium. The two complete reflectances $\underline{R}$ are obtained algebraically from the complete transmittances T and the slab reflectances determined in preliminary sweeps. The distinguishing feature of the present method (and its dual companion to be considered in section 21) is the stability of the numerical integration in solving for the two flows of the light field. This is in contrast to the basically unstable solution procedure afforded by the fundamental matrix method of section 14. The price for attaining this stability is paid in the form of preliminary sweeps of the medium that must be made to determine the appropriate one of the two (upward and downward) standard reflectances $\underline{\mathrm{R}}$ and one of the two standard source-generated irradiances $\underline{\mathrm{H}}_\eta$ of the medium. These results will serve as initial values to start the $\underline{T}$ and $\underline{H}_\eta$ integrations. Then, as the transport equations for $\underline{T}$ and $\underline{H}_\eta$ are integrated in reverse sweeps over the medium, these established $\underline{\mathrm{R}}$ and $\underline{\mathrm{H}}_\eta$ values are regenerated and used to guide the integrations in a stable way.

The medium-partitioning, the given data, and the boundary conditions for the present procedure are given in section 19A, and will be referred to repeatedly in what follows.

B. *The $\underline{\mathcal{T}}$ $\underline{\mathit{H}}_\eta$ Analytic Procedure*

There are six main stages in the present procedure.

I. Integrate the major Riccati pair (16.6), (16.8) in the form

$$
\begin{cases}
\dfrac{\partial \underline{R}(y,a)}{\partial y} = \underline{R}(y,a)\,[\underline{\tau}(y,-) + \underline{\rho}(y,-)\,\underline{R}(y,a)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\,\underline{R}(y,a)] \\[2ex]
\dfrac{\partial \underline{H}_\eta(a,y)}{\partial y} = \underline{H}_\eta(a,y)\,[\underline{\tau}(y,-) + \underline{\rho}(y,-)\,\underline{R}(y,a)] + [\underline{h}_e(y,-) + \underline{h}_e(y,+)\,\underline{R}(y,a)]
\end{cases}
$$

a
x
y
z
b

The integration starts at level x with initial values (ia) (of section 19A) and proceeds downward to level z using (iiia) in order to obtain the values $\underline{R}(z,a)$ and $\underline{H}_\eta(a,z)$. From these we compute

$$
\begin{cases}
\underline{\mathcal{T}}(b,z,a) = \underline{T}(b,z)\,[\underline{I}_m - \underline{R}(z,a)\,\underline{R}(z,b)]^{-1} \\[2ex]
\underline{\mathit{H}}_\eta(b,z,a) = [\underline{H}_\eta(b,z) + \underline{H}_\eta(a,z)\,\underline{R}(z,b)]\,[\underline{I}_m - R(z,a)\,\underline{R}(z,b)]^{-1}
\end{cases}
$$

Here we have used (17.35) and (17.39) in which $\underline{R}(z,b)$, $\underline{T}(b,z)$, and $\underline{H}_\eta(b,z)$ are given in (iia) (of section 19A).

II. Integrate the $\underline{R}$, $\underline{H}_\eta$, $\underline{\mathcal{T}}$, $\underline{\mathcal{H}}_\eta$ quartet

$$\left\{\begin{aligned}
&\frac{d\underline{R}(y,a)}{dy} = R(y,a)\,[\underline{\tau}(y,-) + \underline{\rho}(y,-)\,\underline{R}(y,a)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\,\underline{R}(y,a)] \\
&\frac{\partial \underline{H}_\eta(a,y)}{\partial y} = \underline{H}_\eta(a,y)\,[\underline{\tau}(y,-) + \underline{\rho}(y,-)\,\underline{R}(y,a)]\,[\underline{h}_e(y,-) + \underline{h}_e(y,+)\,\underline{R}(y,a)] \\
&-\frac{\partial \underline{\mathcal{T}}(b,y,a)}{\partial y} = \underline{\mathcal{T}}(b,y,a)\,[\underline{\tau}(y,+) + \underline{R}(y,a)\,\underline{\rho}(y,-)] \\
&-\frac{\partial \underline{\mathcal{H}}_\eta(b,y,a)}{\partial y} = \underline{\mathcal{H}}_\eta(b,y,a)\,[\underline{\tau}(y,+) + \underline{R}(y,a)\,\underline{\rho}(y,-)] + [\underline{H}_\eta(a,y)\,\underline{\rho}(y,-) + \underline{h}_e(y,+)]
\end{aligned}\right.$$

$$\begin{array}{ll} a & \rule{2cm}{0.4pt} \\ x & \rule{2cm}{0.4pt} \\ y & \rule{2cm}{0.4pt} \\ z & \rule{2cm}{0.4pt} \\ b & \rule{2cm}{0.4pt} \end{array}$$

and from $\underline{\mathcal{T}}(b,y,a)$ and $\underline{R}(y,a)$ find

$$\left\{\underline{\mathcal{R}}(b,y,a) = \underline{\mathcal{T}}(b,y,a)\,\underline{R}(y,a)\right.$$

The integration starts at level z with the $\underline{R}(z,a)$, $\underline{H}_\eta(a,z)$ and the $\underline{\mathcal{T}}(b,z,a)$, $\underline{\mathcal{H}}_\eta(b,z,a)$ initial values found in stage I and proceeds upward to level x. Store $\underline{\mathcal{T}}(b,y_j,a)$, $\underline{\mathcal{R}}(b,y_j,a)$, and $\underline{\mathcal{H}}_\eta(b,y_j,a)$, $j = 0,\ldots,n$ as they evolve, where $\{y_j\colon\ j = 0,\ldots,n\}$ is the partition of $X(x,z)$ defined in section 19A. The $\underline{\mathcal{T}}$ and $\underline{\mathcal{H}}_\eta$ differential equations above are given in (17.70) and (17.71), while the $\underline{R}$ and $\underline{H}_\eta$ equations are those used in Stage I. The first two equations are used to generate the evanescent values $\underline{R}(y,a)$, $\underline{H}_\eta(a,y)$ needed to march the second two equations up the water body.

III. Integrate the major Riccati pair (16.14), (16.16) in the form

$$\begin{array}{l} a\ \rule{2em}{0.4pt} \\ x\ \rule{2em}{0.4pt} \\ y\ \rule{2em}{0.4pt} \\ z\ \rule{2em}{0.4pt} \\ b\ \rule{2em}{0.4pt} \end{array}$$

$$\left\{ \begin{aligned} -\frac{\partial \underline{R}(y,b)}{\partial y} &= \underline{R}(y,b)\ [\underline{\tau}(y,+) + \underline{\rho}(y,+)\ \underline{R}(y,b)] + [\underline{\rho}(y,-) + \underline{\tau}(y,-)\ \underline{R}(y,b)] \\ -\frac{\partial \underline{H}_\eta(b,y)}{\partial y} &= \underline{H}_\eta(b,y)\ [\underline{\tau}(y,+) + \underline{\rho}(y,+)\ \underline{R}(y,b)] + [\underline{h}_e(y,+) + \underline{h}_e(y,-)\ \underline{R}(y,b)] \end{aligned} \right.$$

The integration starts at level z with initial values (iia) and proceeds upward to level x using (iiia) in order to obtain the values $\underline{R}(x,b)$ and $\underline{H}_\eta(b,x)$ from which we compute

$$\left\{ \begin{aligned} \underline{\mathcal{T}}(a,x,b) &= \mathrm{T}(a,x)\ [\underline{I}_m - \underline{R}(x,b)\ \mathrm{R}(x,a)]^{-1} \\ \underline{\mathcal{H}}_\eta(a,x,b) &= [\underline{H}_\eta(a,x) + \underline{H}_\eta(b,x)\ \underline{R}(x,a)]\ [\underline{I}_m - \underline{R}(x,b)\ \underline{R}(x,a)]^{-1} \end{aligned} \right.$$

Here we have used (17.37) and (17.40) in which $\underline{R}(x,a)$, $\underline{T}(a,x)$, and $\underline{H}_\eta(a,x)$ are given in (ia).

IV. Integrate the $\underline{R}$, $\underline{H}_\eta$, $\underline{\mathcal{T}}$, $\underline{\mathcal{H}}_\eta$ quartet

$$\left\{ \begin{aligned} -\frac{\partial \underline{R}(y,b)}{\partial y} &= \underline{R}(y,b)\ [\underline{\tau}(y,+) + \underline{\rho}(y,+)\ \underline{R}(y,b)] + [\underline{\rho}(y,-) + \underline{\tau}(y,-)\ \underline{R}(x,b)] \\ -\frac{\partial \underline{H}_\eta(b,y)}{\partial y} &= \underline{H}_\eta(b,y)\ [\underline{\tau}(y,+) + \underline{\rho}(y,+)\ \underline{R}(y,b)] + [\underline{h}_e(y,+) + \underline{h}_e(y,-)\ \underline{R}(y,b)] \\ -\frac{\partial \underline{\mathcal{T}}(a,y,b)}{\partial y} &= \underline{\mathcal{T}}(a,y,b)\ [\underline{\tau}(y,-) + \underline{R}(y,b)\ \underline{\rho}(y,+)] \\ \frac{\partial \underline{\mathcal{H}}_\eta(a,y,b)}{\partial y} &= \underline{\mathcal{H}}_\eta(a,y,b)\ [\underline{\tau}(y,-) + \underline{R}(y,b)\ \underline{\rho}(y,+)] + [\underline{H}_\eta(b,y)\ \underline{\rho}(y,+) + \underline{h}_e(y,-)] \end{aligned} \right.$$

$$\begin{array}{l} a\ \rule{2em}{0.4pt} \\ x\ \rule{2em}{0.4pt} \\ y\ \rule{2em}{0.4pt} \\ z\ \rule{2em}{0.4pt} \\ b\ \rule{2em}{0.4pt} \end{array}$$

and from $\underline{\mathcal{T}}(a,y,b)$ and $\underline{R}(y,b)$ find

$$\left\{ \underline{\mathit{R}}(a,y,b) = \underline{\mathit{T}}(a,y,b)\ \underline{\mathrm{R}}(y,b) \right.$$

The integration starts at level x with the $\underline{\mathrm{R}}(x,b)$, $\underline{\mathrm{H}}_\eta(b,x)$ and $\underline{\mathit{T}}(a,x,b)$, $\underline{\mathit{H}}_\eta(a,x,b)$ initial values found in Stage III and proceeds downward to level z. Store $\underline{\mathit{T}}(a,y_j,b)$, $\underline{\mathit{R}}(a,y_j,b)$, and $\underline{\mathit{H}}_\eta(a,y_j,b)$, $j = 0,\ldots,n$, as they evolve. The $\underline{\mathit{T}}$ and $\underline{\mathit{H}}_\eta$ differential equations are given in (17.65) and (17.66). These two equations are guided in their downward march by the $\underline{\mathrm{R}}(y,b)$, $\underline{\mathrm{H}}_\eta(b,y)$ values supplied immediately before by the first two differential equations.

V. Using (iiib,c) and the stored results of stages II, IV, find, for $j = 0,\ldots,n$,

$$\boxed{\begin{aligned} \underline{\mathrm{H}}(y_j,+) &= \underline{\mathrm{H}}(b,+)\ \underline{\mathit{T}}(b,y_j,a) + \underline{\mathrm{H}}(a,-)\ \underline{\mathit{R}}(a,y_j,b) + \underline{\mathit{H}}_\eta(b,y_j,a) \\ \underline{\mathrm{H}}(y_j,-) &= \underline{\mathrm{H}}(b,+)\ \underline{\mathit{R}}(b,y_j,a) + \underline{\mathrm{H}}(a,-)\ \underline{\mathit{T}}(a,y_j,b) + \underline{\mathit{H}}_\eta(a,y_j,b) \end{aligned}}$$

These results are stored.

VI. Using (ib), (iib) and the results of stages II and IV, compute

$$\left\{\begin{aligned} \underline{\mathrm{T}}(b,a) &= \underline{\mathit{T}}(b,x,a)\ \underline{\mathrm{T}}(x,a) \\ \underline{\mathrm{R}}(a,b) &= \underline{\mathrm{R}}(a,x) + \underline{\mathit{R}}(a,x,b)\ \underline{\mathrm{T}}(x,a) \\ \underline{\mathrm{H}}_\eta(b,a) &= \underline{\mathrm{H}}_\eta(x,a) + \underline{\mathit{H}}_\eta(b,x,a)\ \underline{\mathrm{T}}(x,a) \end{aligned}\right.$$

and also

$$\begin{cases} \underline{T}(a,b) = \underline{\mathcal{T}}(a,z,b)\ \underline{T}(z,b) \\ \underline{R}(b,a) = \underline{R}(b,z) + \underline{\mathcal{R}}(b,z,a)\ \underline{T}(z,b) \\ \underline{H}_\eta(a,b) = \underline{H}_\eta(z,b) + \underline{\mathcal{H}}_\eta(a,z,b)\ \underline{T}(z,b) \end{cases}$$

For these computations use the union rules (18.25)-(18.27) and (18.28)-(18.30).

Recall that $x = y_0$ and $z = y_n$, so that, e.g., $\underline{\mathcal{T}}(b,x,a)$ is $\underline{\mathcal{T}}(b,y_0,a)$ and $\underline{\mathcal{T}}(a,z,b)$ is $\underline{\mathcal{T}}(a,y_n,b)$ as found in stages II and IV, respectively.

The matrices $\underline{R}(a,x)$, $\underline{T}(x,a)$ and vector $\underline{H}_\eta(x,a)$ are given by (ib) while $\underline{R}(b,z)$, $\underline{T}(z,b)$, and $\underline{H}_\eta(z,b)$ are given by (iib). From the above results and (iiib,c) we find

$$\boxed{\begin{aligned} \underline{H}(a,+) &= \underline{H}(b,+)\ \underline{T}(b,a) + \underline{H}(a,-)\ \underline{R}(a,b) + \underline{H}_\eta(b,a) \\ \underline{H}(b,-) &= \underline{H}(b,+)\ \underline{R}(b,a) + \underline{H}(a,-)\ \underline{T}(a,b) + \underline{H}_\eta(a,b) \end{aligned}}$$

C. *Checking the Results*

To check these results, proceed as in section 19C.

D. *General Comments*

The stability of the present method rests in the $\underline{\mathcal{T}}\ \underline{\mathcal{H}}_\eta$ equations of Stages II and IV. A formal proof of the stability rests on the negativity of the matrices $\underline{\tau}(y,+) + \underline{R}(y,a)\ \underline{\rho}(y,-)$ and $\underline{\tau}(y,-) + \underline{R}(y,b)\ \underline{\rho}(y,+)$ in real media. Thus $\underline{\mathcal{T}}(b,y,a)$ (in the upward sweep) and $\underline{\mathcal{T}}(a,y,b)$ (in the downward sweep) tend to be exponentially damped. As the integrations of these equations proceed, any numerical 'glitches' or inaccuracies in $\underline{\mathcal{T}}$ or $\underline{\mathcal{H}}_\eta$ are eventually exponentially damped out. Observe how $\underline{R}(y,a)$ and $\underline{H}_\eta(a,y)$ found in stage I by a downward Riccati sweep are 'deintegrated' in the upward sweep of the same Riccati pair in stage II; and how $\underline{R}(y,b)$ and $\underline{H}_\eta(b,y)$ are deintegrated in a similar manner in the downward sweep of IV.

21. TRANSPORT SOLUTION OF THE LIGHT FIELD: $\underline{R},\underline{H}_\eta$ ANALYTIC PROCEDURE

A. *Introductory Comments*

Finally, we consider the third of three transport solution procedures of the light field. The present procedure is built around the integration of the $\underline{R},\underline{H}_\eta$ pair and is the natural companion to the computation in section 20. It is worth writing out the key parts of the various stages of this procedure because, while it is formally the companion to that in section 20, the basic equations and some other statements differ in rather interesting ways, the full understanding of which comes only after mastering the connections among the complete transfer operators, as developed in section 17. Because of these differences, the relative effectiveness of these three different numerical solution procedures remains open to further study, which will not be done here.

B. *The $\underline{R}$ $\underline{H}_\eta$ Analytic Procedure*

There are six main stages in the present procedure.

I. Integrate the major Riccati pair (16.6), (16.8) in the form

$$\left\{\begin{aligned} \frac{\partial \underline{R}(y,a)}{\partial y} &= \underline{R}(y,a)[\underline{\tau}(y,-) + \underline{\rho}(y,-)\,\underline{R}(y,a)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\,\underline{R}(y,a)] \\ \frac{\partial \underline{H}_\eta(a,y)}{\partial y} &= \underline{H}_\eta(a,y)[\underline{\tau}(y,-) + \underline{\rho}(y,-)\,\underline{R}(y,a)] + [\underline{h}_e(y,-) + \underline{h}_e(y,+)\,\underline{R}(y,a)] \end{aligned}\right. \qquad \begin{array}{l} a \\ x \\ y \\ z \\ b \end{array}$$

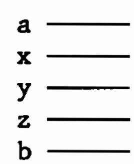
The integration starts at level x with initial values (ia) (of section 9A) and proceeds downward to level z using (iiia) in order to obtain the values $\underline{R}(z,a)$ and $\underline{H}_\eta(a,z)$ from which we compute

$$\begin{cases} \underline{\mathcal{R}}(b,z,a) = \underline{T}(b,z)\,\underline{R}(z,a)\,[\underline{I}_m - \underline{R}(z,b)\,R(z,a)]^{-1} \\ \underline{\mathcal{H}}_\eta(a,z,b) = [\underline{H}_\eta(a,z) + \underline{H}_\eta(b,z)\,\underline{R}(z,a)]\,[\underline{I}_m - \underline{R}(z,b)\,\underline{R}(z,a)]^{-1} \end{cases}$$

The parts $\underline{R}(z,a)$, $\underline{H}_\eta(a,z)$ come from the major Riccati pair's sweep while $R(z,b)$, $T(b,z)$ and $\underline{H}_\eta(b,z)$ come from (iia).

II. Integrate the $\underline{R}$, $\underline{T}$, $\underline{H}_\eta$ major trio (16.14)-(16.16) and the $\underline{\mathcal{R}}$ $\underline{\mathcal{H}}_\eta$ complete pair (17.80), (17.81):

$$\begin{cases} -\dfrac{d\underline{R}(y,b)}{dy} = \underline{R}(y,b)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{R}(y,b)] + [\underline{\rho}(y,-) + \underline{\tau}(y,-)\,\underline{R}(y,b)] \\ -\dfrac{\partial T(b,y)}{\partial y} = \underline{T}(b,y)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{R}(y,b)] \\ -\dfrac{\partial \underline{H}_\eta(b,y)}{\partial y} = \underline{H}_\eta(b,y)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{R}(y,b)] + [\underline{h}_e(y,+) + \underline{h}_e(y,-)\,\underline{R}(y,b)] \\ \dfrac{\partial \underline{\mathcal{R}}(b,y,a)}{\partial y} = \underline{\mathcal{R}}(b,y,a)\,[\underline{\tau}(y,-) + \underline{R}(y,b)\,\underline{\rho}(y,+)] + \underline{T}(b,y)\,\underline{\rho}(y,+) \\ \dfrac{\partial \underline{\mathcal{H}}_\eta(a,y,b)}{\partial y} = \underline{\mathcal{H}}_\eta(a,y,b)[\underline{\tau}(y,-) + \underline{R}(y,b)\,\underline{\rho}(y,+)] + [\underline{H}_\eta(b,y)\,\underline{\rho}(y,+) + \underline{h}_e(y,+)] \end{cases}$$



$$\underline{\mathcal{T}}(b,y,a) = \underline{T}(b,y) + \underline{\mathcal{R}}(b,y,a)\,\underline{R}(y,b)$$

The integration starts at level z with initial values (iia) for $\underline{R}(y,b)$, $\underline{T}(b,y)$, and $\underline{H}_\eta(b,y)$ and with initial values $\underline{\mathcal{R}}(b,z,a)$, $\underline{\mathcal{H}}_\eta(a,z,b)$ found in Stage I. At each level y of interest, $\mathcal{T}(b,y,a)$ is found, as shown, from $\underline{T}(b,y)$, $\underline{\mathcal{R}}(b,y,a)$ and $R(y,b)$ (cf. imbedding rule (17.63)). Store $\mathcal{R}(b,y_j,a)$, $\mathcal{T}(b,y_j,a)$, and $\underline{\mathcal{H}}_\eta(a,y_j,b)$, $j = 0,\ldots,n$.

III. Integrate the major Riccati pair (16.14), (16.16) in the form

$$-\frac{\partial \underline{R}(y,b)}{\partial y} = \underline{R}(y,b)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{R}(y,b)] + [\underline{\rho}(y,-) + \underline{\tau}(y,-)\,\underline{R}(y,b)]$$

$$-\frac{\partial \underline{H}_\eta(b,y)}{\partial y} = \underline{H}_\eta(b,y)[\underline{\tau}(y,+) + \underline{\rho}(y,+)\,\underline{R}(y,b)] + [\underline{h}_e(y,+) + \underline{h}_e(y,-)\underline{R}(y,b)]$$

$$\begin{array}{l} a\ \text{———} \\ x\ \text{———} \\ y\ \text{———} \\ z\ \text{———} \\ b\ \text{———} \end{array}$$

The integration starts at level z with initial values (iia) and proceeds upward to level x using (iiia) in order to obtain the values $\underline{R}(x,b)$ and $\underline{H}_\eta(b,x)$ from which we compute

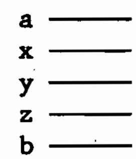
$$\underline{\mathcal{R}}(a,x,b) = \underline{T}(a,x)\,\underline{R}(x,b)[\underline{I}_m - \underline{R}(x,a)\,\underline{R}(x,b)]^{-1}$$

$$\underline{\mathcal{H}}_\eta(b,x,a) + [\underline{H}_\eta(b,x) + \underline{H}_\eta(a,x)\,\underline{R}(x,b)]\,[\underline{I}_m - \underline{R}(x,a)\,R(x,b)]^{-1}$$

The parts $\underline{R}(x,b)$, $\underline{H}_\eta(b,x)$ come from the major Riccati pair's sweep while $\underline{R}(x,a)$, $\underline{T}(a,x)$ and $\underline{H}_\eta(a,x)$ come from (ia).

IV. Integrate the $\underline{R}$, $\underline{T}$, $\underline{H}_\eta$ major trio (16.6)-(16.8) and the $\underline{\mathcal{R}}$ $\underline{\mathcal{H}}_\eta$ complete pair (17.75), (17.76)

$$\left\{\begin{aligned}
&\frac{\partial \underline{R}(y,a)}{\partial y} = \underline{R}(y,a)[\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] + [\underline{\rho}(y,+) + \underline{\tau}(y,+)\ \underline{R}(y,a)] \\
&\frac{\partial \underline{T}(a,y)}{\partial y} = \underline{T}(a,y)[\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] \\
&\frac{\partial \underline{H}_\eta(a,y)}{\partial y} = \underline{H}_\eta(a,y)[\underline{\tau}(y,-) + \underline{\rho}(y,-)\ \underline{R}(y,a)] + [\underline{h}_e(y,-) + \underline{h}_e(y,+)\ \underline{R}(y,a) \\
&-\frac{\partial \underline{\mathcal{R}}(a,y,b)}{\partial y} = \underline{\mathcal{R}}(a,y,b)[\underline{\tau}(y,+) + \underline{R}(y,a)\ \underline{\rho}(y,-)] + \underline{T}(a,y)\ \underline{\rho}(y,-) \\
&-\frac{\partial \underline{\mathcal{H}}_\eta(b,y,a)}{\partial y} = \underline{\mathcal{H}}_\eta(b,y,a)[\underline{\tau}(y,+) + \underline{R}(y,a)\ \underline{\rho}(y,-)] + [\underline{H}_\eta(a,y)\ \underline{\rho}(y,-) + \underline{h}_e(y,+)]
\end{aligned}\right.$$



$$\left\{\ \underline{\mathcal{T}}(a,y,b) = \underline{T}(a,y) + \underline{\mathcal{R}}(a,y,b)\ \underline{R}(y,a)\right.$$

The integration starts at level x with initial values (ia) for $\underline{R}(y,a)$, $\underline{T}(a,y)$, and $\underline{H}_\eta(a,y)$ and with initial values $\mathcal{R}(a,x,b)$, $\underline{\mathcal{H}}_\eta(b,x,a)$ found in Stage III. At each level of interest $\underline{\mathcal{T}}(a,y,b)$ is found as shown (cf. imbedding rule (17.60)). Store $\underline{\mathcal{R}}(a,y_j,b)$, $\mathcal{T}(a,y_j,b)$, and $\underline{\mathcal{H}}_\eta(b,y_j,a)$, $j = 0,\ldots,n$.

V. Using (iiib,c) and the stored results of Stages II, IV, find for $j = 0,\ldots,n$

$$\boxed{\begin{aligned}
\underline{H}(y_j,+) &= \underline{H}(b,+)\ \underline{\mathcal{T}}(b,y_j,a) + \underline{H}(a,-)\ \underline{\mathcal{R}}(a,y_j,b) + \underline{\mathcal{H}}_\eta(b,y_j,a) \\
\underline{H}(y_j,-) &= \underline{H}(b,+)\ \underline{\mathcal{R}}(b,y_j,a) + \underline{H}(a,-)\ \underline{\mathcal{T}}(a,y_j,b) + \underline{\mathcal{H}}_\eta(a,y_j,b)
\end{aligned}}$$

These results are stored.

VI. Using (ib), (iib) and the results of Stages II and IV, compute

$$\begin{cases} \underline{T}(b,a) = \underline{\mathcal{T}}(b,x,a)\ \underline{T}(x,a) \\ \underline{R}(a,b) = \underline{R}(a,x) + \underline{\mathcal{R}}(a,x,b)\ \underline{T}(x,a) \\ \underline{H}_\eta(b,a) = \underline{H}_\eta(x,a) + \underline{\mathcal{H}}_\eta(b,x,a)\ T(x,a) \end{cases}$$

and also

$$\begin{cases} \underline{T}(a,b) = \underline{\mathcal{T}}(a,z,b)\ \underline{T}(z,b) \\ \underline{R}(b,a) = \underline{R}(b,z) + \underline{\mathcal{R}}(b,z,a)\ \underline{T}(z,b) \\ \underline{H}_\eta(a,b) = \underline{H}_\eta(z,b) + \underline{\mathcal{H}}_\eta(a,z,b)\ \underline{T}(z,b) \end{cases}$$

For these computations use the union rules (18.25)-(18.27) and (18.28)-(18.30). From the above results and (iiib,c) we find

$$\boxed{\begin{aligned} \underline{H}(a,+) &= \underline{H}(b,+)\ \underline{T}(b,a) + \underline{H}(a,-)\ \underline{R}(a,b) + \underline{H}_\eta(b,a) \\ \underline{H}(b,-) &= \underline{H}(b,+)\ \underline{R}(b,a) + \underline{H}(a,-)\ \underline{T}(a,b) + \underline{H}_\eta(a,b) \end{aligned}}$$

C. *Checking the Results*

To check these results, proceed as in section 19C.

D. *General comments*

Various dualities and differences in the present $\underline{\mathcal{R}}\ \underline{\mathcal{H}}_\eta$ procedure and the $\underline{\mathcal{T}}\ \underline{\mathcal{H}}_\eta$ procedure in section 20 may be noted. The first part of Stage I in each case is the same, but the $\underline{\mathcal{T}}\ \underline{\mathcal{H}}_\eta$ and $\underline{\mathcal{R}}\ \underline{\mathcal{H}}_\eta$ values derived therefrom are quite different. The Riccati pair in Stage II of Section 20 is now replaced by a

Riccati trio of a different kind (downward type in sec. 20, upward type in the present section). Moreover, while the standard Riccati pair in Stage II of sec. 20 was deintegrated to supply material for the complete pair $\underline{T}\ \underline{H}_\eta$, now it is the complete Riccati pair $\underline{R}\ \underline{H}_\eta$ pair that is being deintegrated back to the x level. Observe how the $\underline{T}$ equation of the major Riccati trio in Stage II of the present section is essential to the $\underline{R}$-equation, while it is missing from Stage II of section 20. Observe finally how the $\underline{R}(y,a)$, $\underline{H}_\eta(a,y)$ pair is integrated downward on two separate occasions in Stages I and IV, while the $\underline{R}(y,b)$, $\underline{H}_\eta(b,y)$ pair is integrated upward on two separate occasions in Stages II and III. The end values of the integration of the Riccati trio in Stage II can be saved to avoid the integration in Stage III. However, the Stage I and Stage IV duplications seem unavoidable.

PART IV. INVERSE SOLUTION OF IRRADIANCE MODEL

22. ASSUMPTIONS UNDERLYING THE INVERSE PROCEDURE AND A TALLY OF UNKNOWNS

We now reverse the procedure of Part III. There the local optical properties $\tau(y,\pm|i,j)$, $\rho(y,\pm|i,j)$ in (11.15) over the y range $x \le y \le z$ were assumed known, and we desired the irradiances $H(y,\pm,j)$ over that range. Now we have the case that the light field components $H(y,\pm,j)$ have been measured throughout the water body and we wish to estimate the local optical properties $\tau(y,\pm|,i,j)$, $\rho(y,\pm|i,j)$ for y in the range $a \le y \le b$. This is the *inverse problem* of the irradiance model. Towards this end the following preliminary steps A-F, involving assumptions and constructions, are entailed.

A. As a first step in the inverse solution procedure, we choose a subinterval $X[x,z]$ from $X[a,b]$ over which we wish to determine the values of $a(y,j)$, $\bar{b}(y,j)$ and $\bar{s}(y,i,j)$, $i,j = 1,\dots,m$, used in the irradiance model of sec. 11. The subinterval $X[x,z]$ is to be taken small enough so that $a(y,j)$, $\bar{b}(y,j)$ and $\bar{s}(y,i,j)$, where $x \le y \le z$, are to be considered constant over it. The subinterval is in turn divided into m-1 subslabs by choosing the m equally spaced partitioning depths y_j, such that $x = y_1 < y_2 < \cdots < y_m = z$. See Fig. 8.

B. The irradiances $H(y_k,\pm,j)$ have been measured at the depths y_k of $X[x,z]$, defined in par. A, and at each depth y_k the measurements for each wavelength index $j = 1,\dots,m$ have been filed for use. It is assumed that in $X[a,b]$ there are no true emissive sources, i.e., $h_e(y,\pm,\lambda) \equiv 0$ in (10.15) for all y.

C. It will be assumed that in each homogeneous subslab $X[x,z]$ of $X[a,b]$ the distribution functions $D(y,\pm,j)$ (cf. (10.4)) are independent of depth y. A

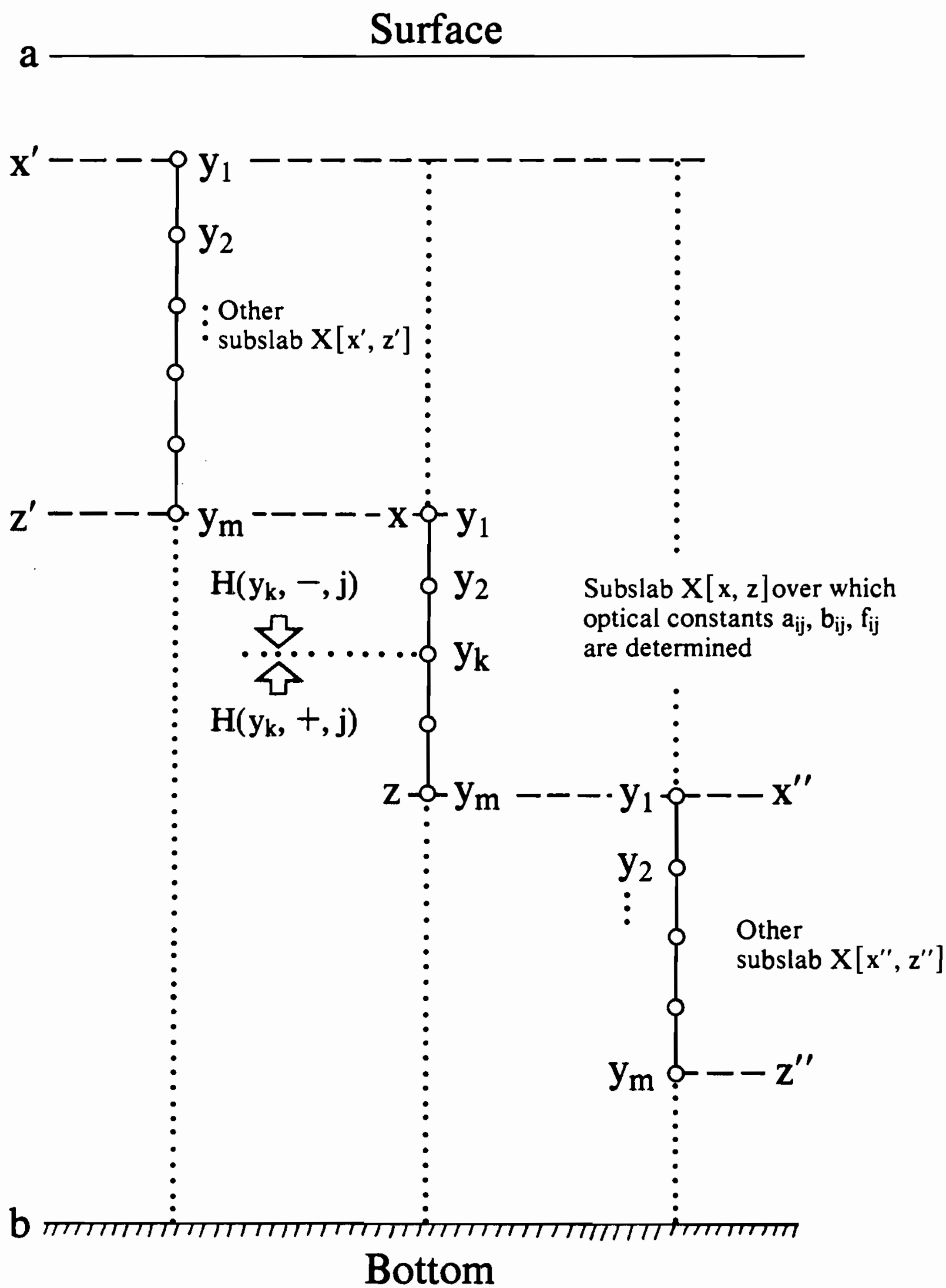


Figure 8--Setting of the irradiance measurements used in the inverse solution of the irradiance model.

similar depth independence assumption holds for the eccentricity functions $\varepsilon_f(y,\pm,i)$, $\varepsilon_b(y,\pm,i)$ in (10.20), (10.21) for depth range $x \le y \le z$, $i = 1,\ldots,m$. $D(y,\pm,j)$, $\varepsilon_f(y,\pm,i)$ $\varepsilon_b(y,\pm,i)$ may take on different values in neighboring subslabs $X[x',z']$, $X[x,z]$,$X[x'',z'']$, etc.; but within each subslab of $X[a,b]$ they are considered constant. Moreover, ε_f and ε_b are assumed independent of the directions (±) of flow.

D. For easy handling of the optical properties during the solution procedure we rewrite the local reflectances $\rho(y,\pm|i,j)$ and local transmittances $\tau(y,\pm|i,j)$ in (11.17) and (11.18) within any given subslab $X[x,z]$ in the following abbreviated notation. First, for the monochromatic properties, write

$$\left.\begin{array}{ll} 'a_{jj}' & \text{for } \bar{a}(y,j) \\ 'b_{jj}' & \text{for } \bar{b}(y,j) \end{array}\right\} \qquad j = 1,\ldots,m \tag{22.1}$$

For the heterochromatic properties write

$$'s_{ij}' \quad \text{for } \bar{s}(y,i,j),\ i,j = 1,\ldots,m,\ i \neq j$$

where $\bar{s}(y,i,j)$ is defined in (11.11) and used in (11.13), for $x \le y \le z$. Next, for the monochromatic properties write

$$\left.\begin{array}{ll} '-D_j(\pm)[a_{jj} + b_{jj}]' & \text{for } \tau(y,\pm|j,j) \\ 'D_j(\pm)\ b_{jj}' & \text{for } \rho(y,\pm|j,j) \end{array}\right\} \qquad j = 1,\ldots,m \tag{22.2}$$

Further, for the heterochromatic properties, write

$$\left.\begin{array}{lll} 'D_i(\pm)\, f_{ij}' & \text{for} & \tau(y,\pm|i,j) \\ & & \\ 'D_i(\pm)\, b_{ij}' & \text{for} & \rho(y,\pm|i,j) \end{array}\right\} \qquad \begin{array}{l} i,j = 1,\ldots,m \\ i \neq j \end{array} \tag{22.3}$$

for $x \le y \le z$. Thus all optical properties are assumed independent of y in $X[x,z]$.

E. Recall that the $D_j(\pm)$ are assumed known within each subslab $X[x,z]$. Hence the unknown properties are the monochromatic absorption and backscatter coefficients a_{jj} and b_{jj}, $j = 1,\ldots,m$, along with their heterochromatic associates b_{ij} and f_{ij}, where in particular (cf. (11.13)):

$$\left.\begin{array}{lll} 'b_{ij}' & \text{stands for} & \varepsilon_b(y,\pm,i)\; s_{ij} \\ & & \\ 'f_{ij}' & \text{stands for} & \varepsilon_f(y,\pm,i)\; s_{ij} \end{array}\right\} \tag{22.4}$$

for $i,j = 1,\ldots,m$, $i \neq j$.

Once the heterochromatic b_{ij} and f_{ij} are determined $i,j = 1,\ldots,m$, $i \neq j$, then the eccentricities $\varepsilon_f(i) \equiv \varepsilon_f(y,\pm|i)$ and $\varepsilon_b(i) \equiv \varepsilon_b(y,\pm|i)$ in each subslab may be found, since the two equations

$$b_{ij}/f_{ij} = \varepsilon_b(i)/\varepsilon_f(i) \qquad \text{(via (22.4))}$$

and

$$1 = \varepsilon_f(i) + \varepsilon_b(i) \qquad \text{(via (10.22))} \tag{22.5}$$

can be solved for the required quantities:

$$\varepsilon_f(i) = [1 + (b_{ij}/f_{ij})]^{-1} \quad , \quad i,j = 1,\ldots,m \tag{22.6}$$

$$\varepsilon_b(i) = 1 - \varepsilon_f(i) \qquad\qquad i \neq j$$

From this, by (22.4), we may find

$$s_{ij} = b_{ij}/\varepsilon_b(i) = f_{ij}/\varepsilon_f(i), \; i \neq j \tag{22.7}$$

which is the transpectral total volume scattering function, the heart of $\tau(y,\pm|i,j)$ and $\rho(y,\pm|i,j)$ in (11.13). Finally, from (22.2) we can determine b_{jj} and hence a_{jj} for $j = 1,\ldots,m$.

F. In summary, then, a tally of the unknown coefficients to be determined shows that in $X[x,z]$ we have $2m^2$ unknowns in all, consisting of (cf. (11.17), (11.18))

(i) *$m^2 + m$ local transmittances* $\tau(y,\pm|i,j)$ consisting of

- m monochromatic absorption coefficients a_{jj}
- m monochromatic backscatter coefficients b_{jj}
- $m(m-1)$ heterochromatic forward scatter coefficients f_{ij}, $i \neq j$

  [totalling $m^2 + m$ distinct unknown coefficients]

(ii) *$m^2 - m$ local reflectances* $\rho(y,\pm|i,j)$ consisting of

- m monochromatic backscatter coefficients b_{jj} (the same ones as in (i) and therefore not counted)
- $m(m-1)$ heterochromatic backward scatter coefficients b_{jj}, $i \neq j$

  [totalling $m^2 - m$ distinct unknown coefficients]

Using the $2m^2$ pieces of irradiance information $\{H(y_k,\pm,j): \;\; j,k = 1,\ldots,m\}$ we will next show how to determine the $2m^2$ unknowns listed above.

23. PRIMING THE PUMP: FINDING THE MONOCHROMATIC UNKNOWNS

The full inverse procedure, defined in sec. 24, will be given a good start if we can hand it some initial estimates of the a_{ij}, b_{ij}, and f_{ij} in each subslab X[x,z] of X[a,b]. This will shorten the iterative work which the inverse procedure must do to home in on the true values of these coefficients. In this section we will develop a monochromatic version of the full procedure that will serve this purpose. The end result will give estimates of the monochromatic quantities defined by a_{jj} and b_{jj} in (22.1).

We return to the basic equation (10.15) and shut off the coupling of the wavelength-dependent transfer functions. This in effect drops the integral term over the spectrum Λ and so, with $h_e(y,\pm,\lambda) \equiv 0$ (and with assumption B in sec. 22 in force), we obtain the classic monochromatic irradiance model:

$$\mp \frac{dH(y,\pm)}{dy} = - [a(y,\pm) + b(y,\pm)] \, H(y,\pm) + b(y,\mp) \, H(y,\mp) \tag{23.1}$$

For simplicity of notation, the λ has been dropped and is understood to be one of the m discrete wavelengths in the spectral interval $[\lambda_a,\lambda_b]$ defined in sec. 11. The same preparations gone through to find (11.15) apply automatically to (23.1). Hence following the procedure in Preisendorfer and Mobley (1984), we may then write (23.1) as

$$\mp \frac{dH(y,\pm)}{dy} = D_{\pm}[a+\bar{b}] \, H(y,\pm) + D_{\mp}\bar{b} \, H(y,\mp) \tag{23.2}$$

If we write this in the notation established in (22.1), then we would associate $D_{\pm}$ to $D_j(\pm)$, a to a_{jj} and $\bar{b}$ to b_{jj} for the jth wavelength band of interest (with j understood, for brevity). In what follows, then, we will show how to determine estimates a and $\bar{b}$ of a_{jj} and b_{jj}, respectively from knowledge of $D_{\pm}$ and measurements of $H(y,\pm,j)$. This will result in a numerical exercise gone through separately for each wavelength index $j = 1,\ldots,m$. It should be noted that the a and $\bar{b}$ so found are only approximations of their associated coefficients a_{jj} and b_{jj} in the full model. For if fluorescence is active in the hydrosol, then (23.1) must have on its right side a positive source term: $h_e(y,\pm,\lambda) > 0$ in order to correctly account for the arrival of photons of wavelength λ having been transpectrally scattered from some wavelength $\lambda' \neq \lambda$. It may be noted here that one of the main motivations for the present study is the desire to solve the inverse problem for (23.1) when $h_e(y,\pm,\lambda)$ is positive, due to fluorescence.

The procedure described below is closely patterned after that in Preisendorfer and Mobley (1984). The salient difference is that the present development is based on the *fundamental solution procedure*, while that in the cited reference uses the transport procedure. This is done mainly to explore possible alternate inverse procedures. The cited solution is applicable here also, if one wishes to use it.

A. In Fig. 9 we have two pairs of measured irradiances $H(x,\pm)$, $H(y,\pm)$ at depths x and y in the source-free hydrosol. These pairs are related by means of the mapping property (14.13):

$$\begin{aligned} H(y,+) &= H(x,+)\, \mathcal{M}_{++}(x,y) + H(x,-)\, \mathcal{M}_{-+}(x,y) \\ H(y,-) &= H(x,+)\, \mathcal{M}_{+-}(x,y) + H(x,-)\, \mathcal{M}_{--}(x,y) \end{aligned} \tag{23.3}$$

for $x \le y \le z$.

Now, if we have fixed a and $\bar{b}$ only approximately, then the $\underline{\mathcal{M}}(x,y)$ matrix used in (23.3) may not be exactly computed. When $H(x,\pm)$ are applied to such a

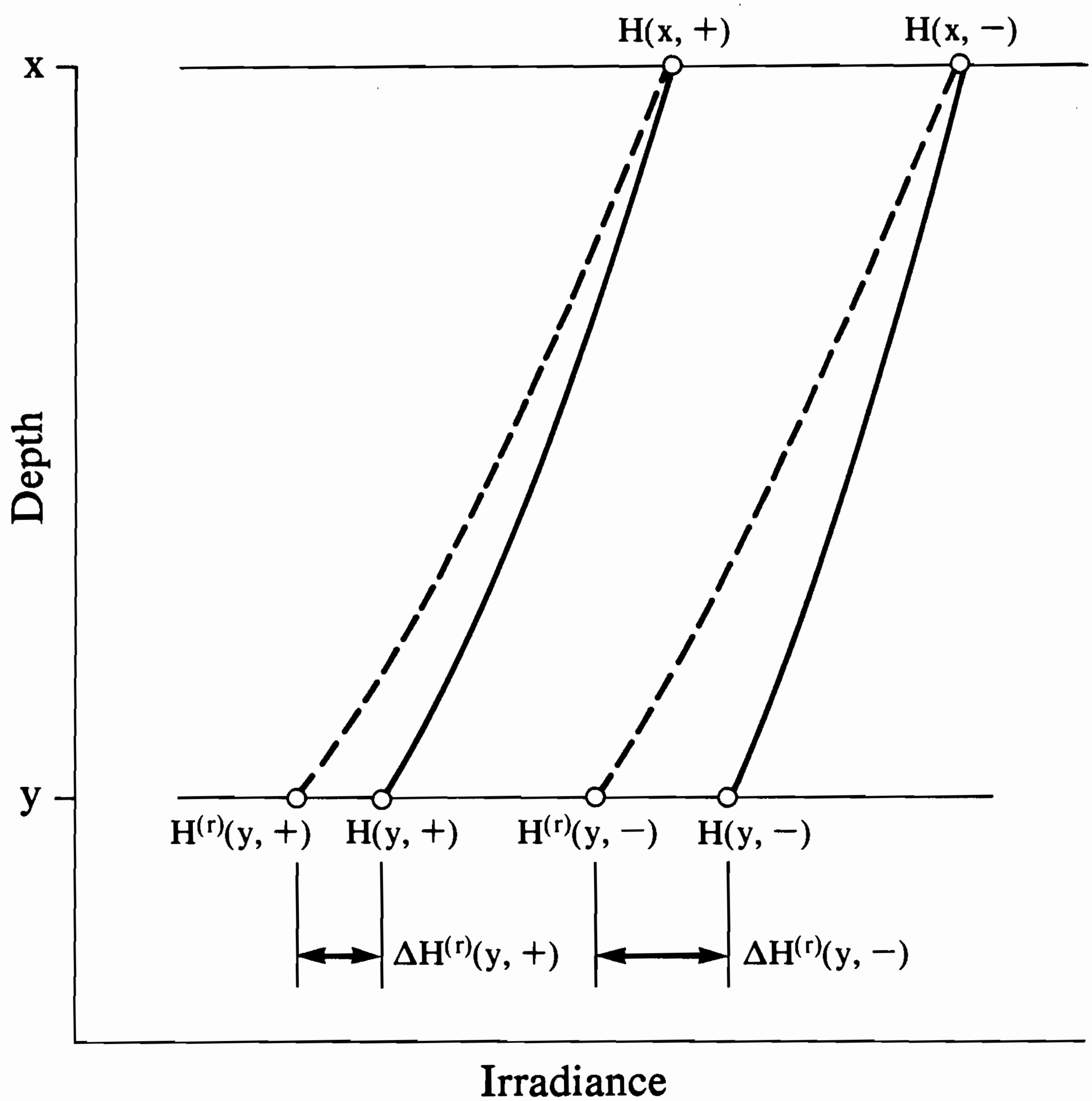


Figure 9--Setting of the iterative procedure used in inverting the irradiance model.

matrix we may produce irradiances $H^{(r)}(y,\pm)$ at level y that differ from the true value $H(y,\pm)$ there by the amounts $\Delta H^{(r)}(y,\pm)$ shown in Fig. 9. This suggests that, in order to correct these errors, we trace in detail the effects on the computed irradiance field of the errors Δa and $\Delta\bar{b}$ in the values of a and $\bar{b}$. Hence consider the differentials $\Delta H(y,\pm)$ given by the calculus:

$$\begin{aligned}\Delta H(y,+) &= \frac{\partial H(y,+)}{\partial a}\,\Delta a + \frac{\partial H(y,+)}{\partial \bar{b}}\,\Delta\bar{b} \\ \Delta H(y,-) &= \frac{\partial H(y,-)}{\partial a}\,\Delta a + \frac{\partial H(y,-)}{\partial \bar{b}}\,\Delta\bar{b}\end{aligned} \tag{23.4}$$

B. From (23.3) we know in principle how to find the derivatives of the irradiance field in (23.4):

$$\begin{aligned}\frac{\partial H(y,+)}{\partial a} &= H(x,+)\,\frac{\partial M_{++}(x,y)}{\partial a} + H(x,-)\,\frac{\partial M_{-+}(x,y)}{\partial a} \\ \frac{\partial H(y,+)}{\partial \bar{b}} &= H(x,+)\,\frac{\partial M_{++}(x,y)}{\partial \bar{b}} + H(x,-)\,\frac{\partial M_{-+}(x,y)}{\partial \bar{b}}\end{aligned} \tag{23.5}$$

and also

$$\begin{aligned}\frac{\partial H(y,-)}{\partial a} &= H(x,+)\,\frac{\partial M_{+-}(x,y)}{\partial a} + H(x,-)\,\frac{\partial M_{--}(x,y)}{\partial a} \\ \frac{\partial H(y,-)}{\partial \bar{b}} &= H(x,+)\,\frac{\partial M_{+-}(x,y)}{\partial \bar{b}} + H(x,-)\,\frac{\partial M_{--}(x,y)}{\partial \bar{b}}\end{aligned} \tag{23.6}$$

For example, knowing $a,\bar{b}$ we can compute the difference quotients for any pair δa, $\delta\bar{b}$ of increments of a and $\bar{b}$ that will approximate the derivation of the $\underline{M}(x,y)$ entries:

$$\frac{\partial M_{++}(x,y;a,\bar{b})}{\partial a} \simeq [M_{++}(x,y;a+\tfrac{1}{2}\delta a,\bar{b}) - M_{++}(x,y;a-\tfrac{1}{2}\delta a,\bar{b})]/\delta a$$

$$\frac{\partial M_{++}(x,y;a,\bar{b})}{\partial \bar{b}} \simeq [M_{++}(x,y;a;\bar{b}+\tfrac{1}{2}\delta\bar{b}) - M_{++}(x,y;a,\bar{b}-\tfrac{1}{2}\delta\bar{b})]/\delta\bar{b} \tag{23.7}$$

In a similar way the remaining six derivatives of the M-functions can be found.

C. The inversion of the system of equations (23.4) is possible if we know the values of the irradiance increments and the four irradiance derivatives evaluated at some pair of values $a,\bar{b}$. First we write (23.4) as

$$\begin{aligned}[\Delta H(y,+),\ \Delta H(y,-)] &= [\Delta a,\Delta\bar{b}] \begin{bmatrix} \dfrac{\partial H(y,+)}{\partial a} & \dfrac{\partial H(y,-)}{\partial a} \\[2ex] \dfrac{\partial H(y,+)}{\partial \bar{b}} & \dfrac{\partial H(y,-)}{\partial \bar{b}} \end{bmatrix} \\ &\equiv [\Delta a,\Delta\bar{b}]\ \underline{C} \end{aligned} \tag{23.8}$$

Then we can find

$$[\Delta a,\Delta\bar{b}] = [\Delta H(y,+),\ \Delta H(y,-)]\ \underline{C}^{-1} \tag{23.9}$$

D. In order to start the monochromatic inverse procedure we need initial estimates of a and $\bar{b}$. For this we use the expressions $K(y,\pm)$ in (12.15), (12.16). In the present homogeneous subslab geometry, these equations can be written, using the notation adopted above, as

$$K(-) = (a+\bar{b})\, D_- - \bar{b} D_+ R(-)$$
$$-K(+) = (a+\bar{b})\, D_+ - \bar{b} D_- R(+) \tag{23.10}$$

Solving these equations for $a+\bar{b}$ and $\bar{b}$, and appending superscripts '0' to denote the fact that these are initial estimates, we have

$$a^{(0)} + \bar{b}^{(0)} = [D_- R^{(+)} K(-) - D_+ R(-)\, K(+)]/\mathrm{Den}$$
$$\bar{b}^{(0)} = [D_- K(+) + D_+ K(-)]/\mathrm{Den} \tag{23.11}$$

where we have written

$$\text{'Den'} \quad \text{for} \quad D_-^2\, R(+) - D_+^2\, R(-) \tag{23.12}$$

Thus once $a^{(0)} + \bar{b}^{(0)}$ and $\bar{b}^{(0)}$ have been determined as in (23.11), we obtain $a^{(0)}$ from the difference $[a^{(0)} + \bar{b}^{(0)}] - \bar{b}^{(0)}$.

The $K(\pm)$ in (23.11) can be estimated using (12.14), thus

$$K(\pm) \approx \frac{-1}{(y-x)} \log_e \frac{H(y,\pm)}{H(x,\pm)} \tag{23.13}$$

The $R(\pm)$ in (23.11) are estimable by

$$R(\pm) \approx \frac{H(x,\mp) + H(z,\mp)}{H(x,\pm) + H(x,\pm)} \tag{23.14}$$

With these approximations $R(\pm)$ and $K(\pm)$ are associated with the mid-depth of $X[x,y]$, i.e., $\frac{1}{2}(x+y)$. These estimates will be practically valid if $(y-x)\alpha$ is on the order of 10^{-1} or less.

E. The general iterative cycle in the monochromatic inverse procedure will now be described. Suppose we are working in subslab X[x,z]. Suppose also that we have gone through r cycles of the iterative process and have at hand estimates $a^{(r)}$ and $\bar{b}^{(r)}$, r = 0,1,.... Then the next [the (r+1)st] cycle is performed in the following sequence of eight steps:

(i) From $a^{(r)}$ and $\bar{b}^{(r)}$ find the rth estimate of the fundamental functions $M_{++}^{(r)}(x,y)$, $M_{+-}^{(r)}(x,y)$, $M_{-+}^{(r)}(x,y)$, $M_{--}^{(r)}(x,y)$ in X[x,z] by integrating over X[x,z] the 2×2 system of equations (cf. (14.4)):

$$\frac{d}{dy}\,\underline{M}^{(r)}(x,y) = \underline{M}^{(r)}(x,y)\,\underline{K}^{(r)} \quad , \quad x \le y \le z \tag{23.15}$$

where

$$\underline{K}^{(r)} = \begin{bmatrix} -\tau_{+}^{(r)} & \rho_{+}^{(r)} \\ -\rho_{-}^{(r)} & \tau_{-}^{(r)} \end{bmatrix} ,$$

$$\underline{M}^{(r)}(x,y) = \begin{bmatrix} M_{++}^{(r)}(x,y) & M_{+-}^{(r)}(x,y) \\ M_{-+}^{(r)}(x,y) & M_{--}^{(r)}(x,y) \end{bmatrix}$$

and where

$$\tau_{\pm}^{(r)} = -D_{\pm}[a^{(r)} + \bar{b}^{(r)}] \quad , \quad \rho_{\pm}^{(r)} = D_{\pm}\bar{b}^{(r)}$$

The expanded form of (23.15) for integration is (the iteration superscript is implicit):

$$\left.\begin{aligned} -\frac{dM_{++}(x,y)}{dy} &= M_{++}(x,y)\ \tau_+ + M_{+-}(x,y)\rho_- \\ \frac{dM_{+-}(x,y)}{dy} &= M_{++}(x,y)\ \rho_+ + M_{+-}(x,y)\tau_- \end{aligned}\right\} \tag{23.16a}$$

$$\left.\begin{aligned} -\frac{dM_{-+}(x,y)}{dy} &= M_{-+}(x,y)\ \tau_+ + M_{--}(x,y)\rho_- \\ \frac{dM_{--}(x,y)}{dy} &= M_{-+}(x,y)\ \rho_+ + M_{--}(x,y)\tau_- \end{aligned}\right\} \tag{23.16b}$$

The initial values for the integration are

$$\begin{aligned} M_{++}(x,x) &= M_{--}(x,x) = 1 \\ M_{+-}(x,x) &= M_{-+}(x,x) = 0 \end{aligned} \tag{23.16c}$$

(ii) Find the rth estimate of the response irradiances $H^{(r)}(y,\pm)$ using (23.3) in the form

$$\begin{aligned} H^{(r)}(y,+) &= H(x,+)\ M_{++}^{(r)}(x,y) + H^{(r)}(x,-)\ M_{-+}^{(r)}(x,y) \\ H^{(r)}(y,-) &= H(x,+)\ M_{+-}^{(r)}(x,y) + H^{(r)}(x,-)\ M_{--}^{(r)}(x,y) \end{aligned} \tag{23.17}$$

Here $H(x,\pm)$ are the measured irradiances of level $x = y_1$ in $X[x,y]$.

(iii) Define the rth estimate of the irradiance increments

$$\Delta H^{(r)}(y,\pm) = H(y,\pm) - H^{(r)}(y,\pm) \tag{23.18}$$

As shown in Fig. 9, these $\Delta H^{(r)}(y,\pm)$ give the discrepancy between the measured irradiance field at level y and the estimated field there.

(iv) Compute the eight derivatives of $\mathcal{M}_{\pm\pm}(x,y)$, $\mathcal{M}_{\mp\pm}(x,y)$ with respect to a and $\bar{b}$, as illustrated by (23.7) and evaluate these derivatives at the rth estimates $a = a^{(r)}$, $\bar{b} = \bar{b}^{(r)}$.

(v) From the eight rth estimates of the $\mathcal{M}$-functions found in (iv), find the four irradiance derivatives $\partial H(y,+)/\partial a,\ldots,\partial H(y,-)/\partial \bar{b}$ as in (23.5), (23.6).

(vi) From the four irradiance derivatives found in (v) and the two irradiance increments found in (iii), use (23.9) to find $\Delta a^{(r+1)}$ and $\Delta \bar{b}^{(r+1)}$.

(vii) Define

$$
\begin{aligned}
a^{(r+1)} &\equiv a^{(r)} + \Delta a^{(r+1)} \\
\bar{b}^{(r+1)} &\equiv \bar{b}^{(r)} + \Delta \bar{b}^{(r+1)}
\end{aligned}
\qquad (23.19)
$$

(viii) Make the tests

$$
\begin{aligned}
|\Delta a^{(r+1)}/a^{(r+1)}| &< \varepsilon \\
|\Delta \bar{b}^{(r+1)}/\bar{b}^{(r+1)}| &< \varepsilon
\end{aligned}
\qquad (23.20)
$$

where ε is some suitably small number, say 10^{-3}.

If this test is not satisfied, return to step (i) and repeat the cycle ending with (23.20). If this test is satisfied, then set $a \equiv a^{(r+1)}$, $\overline{b} \equiv \overline{b}^{(r+1)}$ and go on to the next subslab, beyond X[x,y]. Repeat the initialization step in D and the preceding seven iteration steps.

24. THE GENERAL INVERSE PROCEDURE FOR THE HETEROCHROMATIC UNKNOWNS

We turn next to the basic equation (11.15) and, following the assumptions and constructions in A-F of sec. 22, we are ready to solve for the optical constants a_{ij}, b_{ij}, and f_{ij} representing the absorption, backscatter, and forward scatter properties of the medium $X[x,z]$. These constitute the τ's and ρ's in (11.17) and (11.18). It will be convenient to 'linearize' these two-dimensional arrays as follows. Write

$$'A_\ell' \text{ for } \begin{cases} f_{jk} & \text{if } j \neq k \\ a_{jj} & \text{if } j = k \end{cases} \tag{24.1}$$

where $\ell = (j-1)m+k$, and $j,k = 1,\ldots,m$. Thus, e.g., if $m = 5$, then f_{43}, where now $j = 4$, $k = 3$, is denoted by 'A_{18}' since $\ell = (4-1)\times 5+3 = 18$. Conversely, given A_ℓ, we can find which f_{jk} or a_{jj} it belongs to by dividing ℓ by m. The integer part of the quotient will be $j-1$. Then $k = \ell-(j-1)m$. If $j \neq k$, then A_ℓ is f_{jk}; if $j = k$, then A_ℓ is a_{jj}. Next, we write

$$'B_\ell' \quad \text{for} \quad b_{jk} \tag{24.2}$$

where $\ell = (j-1)m+k$ and $j,k = 1,\ldots,m$. Hence ℓ runs from 1 to m^2 in each case, and the $2m^2$ unknowns to be found are A_ℓ, B_ℓ, $\ell = 1,\ldots,m^2$. These are assembled in the $1\times m^2$ vectors $\underline{A} \equiv [A_1,\ldots,A_{m^2}]$ and $\underline{B} \equiv [B_1,\ldots,B_{m^2}]$.

A. We next find the new multidimensional counterparts to the set (23.3)-(23.9). Thus (23.3), by virtue of the source-free forms of (14.20, 14.21), for each depth y_k, $k = 1,\ldots,m$ in $X[x,z]$ we have

$$\underline{H}(y_k,+) = \underline{H}(x,+)\ \underline{M}_{++}(x,y_k) + \underline{H}(x,-)\ \underline{M}_{-+}(x,y_k)$$
$$\underline{H}(y_k,-) = \underline{H}(x,+)\ \underline{M}_{+-}(x,y_k) + \underline{H}(x,-)\ \underline{M}_{--}(x,y_k) \tag{24.3}$$

where the source terms $\underline{H}_e(x,y,\pm)$ are now zero, by assumption. In numerical calculations it is necessary to write out (24.3) in component form. For example, using the notation (11.16) and denoting the (ij)th components of $\underline{M}_{++}(x,y_k)$ by '$M_{++}(x,y_k|i,j)$', with like notation for the other three matrices, the first of (24.3) is expanded to

$$H(y_k,+,j) = \sum_{i=1}^{m} H(x,+,i)\ M_{++}(x,y_k|i,j) + \sum_{i=1}^{m} H(x,-,i)\ M_{-+}(x,y_k|i,j)$$
$$j,k = 1,\dots,m;\ x = y_1,\ z = y_m \tag{24.3a}$$

and the second of (24.3) is

$$H(y_k,-,j) = \sum_{i=1}^{m} H(x,+,i)\ M_{+-}(x,y_k|i,j) + \sum_{i=1}^{m} H(x,-,i)\ M_{--}(x,y|i,j)$$
$$j,k = 1,\dots,m;\ x = y_1,\ z = y_m \tag{24.3b}$$

The present counterpart to (23.4) is then

$$\Delta H(y_k,\pm,j) = \sum_{\ell=1}^{m^2} \frac{dH(y_k,\pm,j)}{\partial A_\ell}\ \Delta A_\ell + \sum_{\ell=1}^{m^2} \frac{\partial H(y_k,\pm,j)}{\partial B_\ell}\ \Delta B_\ell \tag{24.4a}$$

This system of equations (24.4a) can be placed into matrix form suitable for solving for the A_i, B_j. Thus write

$$\begin{array}{lll} \text{'}\underline{H}(\pm)\text{' for } [\underline{H}(y_1,\pm),\ldots,\underline{H}(y_m,\pm)] & (1\times m^2) & \\ \text{'}\Delta\underline{H}(\pm)\text{' for } [\Delta\underline{H}(y_1,\pm),\ldots,\Delta\underline{H}(y_m,\pm)] & (1\times m^2) & (24.4b) \\ \text{'}\Delta\underline{A}\text{' for } [\Delta A_1,\ldots,\Delta A_{m^2}] & (1\times m^2) & \\ \text{'}\Delta\underline{B}\text{' for } [\Delta B_1,\ldots,\Delta B_{m^2}] & (1\times m^2) & \end{array}$$

where $y_1,\ldots,y_m$ are the partition depths of $X[x,z]$ established in sec. 22A. Moreover, we define an $m^2\times m^2$ matrix $\partial\underline{H}(\pm)/\partial\underline{A}$ such that its element in row p and column q is given by

$$\left[\frac{\partial\underline{H}(\pm)}{\partial\underline{A}}\right]_{pq} \equiv \frac{\partial H(y_k,\pm,j)}{\partial A_p} \qquad (m^2\times m^2) \qquad (24.4c)$$

where $q = (k-1)m+j$ for $j,k = 1,\ldots,m$. Similarly we set

$$\left[\frac{\partial\underline{H}(\pm)}{\partial\underline{B}}\right]_{pq} \equiv \frac{\partial H(y_k,\pm,j)}{\partial B_p} \qquad (m^2\times m^2) \qquad (24.4d)$$

where $q = (k-1)m+j$, for $j,k = 1,\ldots,m$. The range of p and q is $1,\ldots,m^2$. Then (24.4a) can be written as

$$\Delta\underline{H}(\pm) = \Delta\underline{A}\,\frac{\partial\underline{H}(\pm)}{\partial\underline{A}} + \Delta\underline{B}\,\frac{\partial\underline{H}(\pm)}{\partial\underline{B}} \qquad (24.4)$$

which is the heterochromatic counterpart to (23.4).

B. The general form of (23.5) is, for $\ell=1,\ldots,m^2$,

$$\begin{aligned} \frac{\partial\underline{H}(y_k,+)}{\partial A_\ell} &= \underline{H}(x,+)\,\frac{\partial\underline{M}_{++}(x,y_k)}{\partial A_\ell} + \underline{H}(x,-)\,\frac{\partial\underline{M}_{-+}(x,y_k)}{\partial A_\ell} \\ \frac{\partial\underline{H}(y_k,+)}{\partial B_\ell} &= \underline{H}(x,+)\,\frac{\partial\underline{M}_{++}(x,y_k)}{\partial B_\ell} + \underline{H}(x,-)\,\frac{\partial\underline{M}_{-+}(x,y_k)}{\partial B_\ell} \end{aligned} \qquad (24.5)$$

and also

$$\frac{\partial \underline{H}(y_k,-)}{\partial A_\ell} = \underline{H}(x,+)\,\frac{\partial \underline{M}_{+-}(x,y_k)}{\partial A_\ell} + \underline{H}(x,-)\,\frac{\partial \underline{M}_{--}(x,y_k)}{\partial A_\ell}$$

$$\frac{\partial \underline{H}(y_k,-)}{\partial B_\ell} = \underline{H}(x,+)\,\frac{\partial \underline{M}_{+-}(x,y_k)}{\partial B_\ell} + \underline{H}(x,-)\,\frac{\partial \underline{M}_{--}(x,y_k)}{\partial B_\ell} \qquad (24.6)$$

In numerical calculations each of these is opened up and written out as in (24.3a,b). Equation (23.7) now becomes, for $i,j = 1,\ldots,m^2$; $k = 1,\ldots,m$:

$$\frac{\partial \underline{M}_{++}(x,y_k;A_i,B_j)}{\partial A_i} \simeq [\underline{M}_{++}(x,y_k;A_i + \tfrac{1}{2}\delta A_i,B_j) - \underline{M}_{++}(x,y_k;A_i - \tfrac{1}{2}\delta A_i,B_j)]/\delta A_i$$

$$\frac{\partial \underline{M}_{++}(x,y_k;A_i,B_j)}{\partial B_j} \simeq [\underline{M}_{++}(x,y_k;A_i,B_j + \tfrac{1}{2}\delta B_j) - \underline{M}_{++}(x,y_k;A_i,B_j - \tfrac{1}{2}\delta B_j)]/\delta B_j \qquad (24.7)$$

The derivatives of the remaining three $m^2 \times m^2$ block matrices $\underline{M}_{+-}(x,y_k)$, $\underline{M}_{-+}(x,y_k)$, $\underline{M}_{--}(x,y_k)$ in $\underline{M}(x,y_k)$ (cf. (14.3)) are found in an exactly similar way.

C. The inversion of the system of equations (24.4) is in principle possible if we know the values of the irradiance vector's increments $\Delta\underline{H}(\pm)$ and the $4m^4$ irradiance derivatives in (24.4c,d) evaluated at some set of values A_ℓ, B_ℓ, $\ell = 1,\ldots,m^2$. First we write (24.4a) as

$$[\Delta\underline{H}(+),\Delta\underline{H}(-)] = [\Delta\underline{A},\Delta\underline{B}] \begin{bmatrix} \dfrac{\partial \underline{H}(+)}{\partial \underline{A}} & \dfrac{\partial \underline{H}(-)}{\partial \underline{A}} \\[2ex] \dfrac{\partial \underline{H}(+)}{\partial \underline{B}} & \dfrac{\partial \underline{H}(-)}{\partial \underline{B}} \end{bmatrix}$$

That is

$$[\Delta\underline{H}(+),\Delta\underline{H}(-)] \equiv [\Delta\underline{A},\Delta\underline{B}]\ \underline{C} \tag{24.8}$$

Then we can find

$$[\Delta\underline{A},\Delta\underline{B}] = [\Delta\underline{H}(+),\Delta\underline{H}(-)]\ \underline{C}^{-1} \tag{24.9}$$

The inverse of the $2m^2 \times 2m^2$ matrix $\underline{C}$ for, say, $m = 10$ wavelengths over the visible spectrum (so that each wavelength band is 30 nm) requires inversion of a 200×200 matrix. This is easily done on present-day general purpose computers. The question of realizability of the inverse solution therefore devolves on the quality of the measured irradiance field.

D. To start the heterochromatic inverse procedure over the slab $X[x,z]$ we use the initial values a_{jj}, b_{jj}, $j = 1,\ldots,m$ found in sec. 23. The remaining initial values are to be $a_{ij} = 0$, $b_{ij} = 0$, for $i \neq j$ with $i,j = 1,\ldots,m$ and $f_{ij} = 0$ for $i,j = 1,\ldots,m$. Hence the initial $\underline{\rho}$ and $\underline{\tau}$ matrices in (11.17), (11.18) are diagonal matrices with the diagonal elements made up of $\tau(y,\pm|j,j)$ and $\rho(y,\pm|j,j)$ given in (22.2). If there were no fluorescence in the medium, then the work of sec. 23 will already have solved the inverse problem. In a natural fluorescence setting, the off-diagonal elements of the $\underline{\rho}$ and $\underline{\tau}$ matrices are relatively small compared to the diagonal elements, and so the present set of initial values is already well on its way toward the final set.

E. The general iterative cycle in the inverse procedure will now be described. Suppose, as in sec. 23E, we are in slab $X[x,z]$. Suppose also that we have gone through r cycles of the iterative process, $r = 0,1,\ldots$. Then the next, i.e., the $(r+1)$st, cycle is performed in the following sequence of eight steps.

(i) From $\underline{A}^{(r)},\underline{B}^{(r)}$ find the rth estimate of the fundamental $m^2\times m^2$ block matrices $\underline{\mathcal{M}}_{++}^{(r)}(x,y_k)$, $\underline{\mathcal{M}}_{+-}^{(r)}(x,y_k)$, $\underline{\mathcal{M}}_{-+}^{(r)}(x,y_k)$, $\underline{\mathcal{M}}_{--}^{(r)}(x,y_k)$, for $k = 1,\ldots,m$ in $X[x,z]$ by integrating the set of equations (14.4) from level x to level z and storing these matrices at the m depths y_k. The initial values for these matrices are analogous to (14.6), now for dimension $m^2\times m^2$ rather than m. Thus

$$\begin{aligned} \underline{\mathcal{M}}_{++}^{(r)}(x,x) &= \underline{\mathcal{M}}_{--}^{(r)}(x,x) = \underline{I}_{m^2} \\ \underline{\mathcal{M}}_{+-}^{(r)}(x,x) &= \underline{\mathcal{M}}_{-+}^{(r)}(x,x) = \underline{0}_{m^2} \end{aligned} \qquad (24.10)$$

(ii) Find by (24.3) the rth estimate of the response irradiances $\underline{H}^{(r)}(y_k,\pm)$ using the given measured $\underline{H}(y_k,\pm)$ at $k = 1,\ldots,m$ and the $\underline{\mathcal{M}}^{(r)}$ matrices of (i) above.

(iii) Define the rth estimate of the irradiance increments

$$\Delta\underline{H}^{(r)}(y_k,\pm) = \underline{H}(y_k,\pm) - \underline{H}^{(r)}(y_k,\pm) \quad , \quad k = 1,\ldots,m \qquad (24.11)$$

where the $1\times m$ vectors $\underline{H}(y_k,\pm)$ are known from measurements at level y_k, $k = 1,\ldots,m$ (see Fig. 8).

(iv) Compute the A_ℓ, B_ℓ derivatives of the elements of $\underline{M}_{\pm\pm}(x,y_k)$, $\underline{M}_{\pm\mp}(x,y_k)$ (displayed in (24.5), (24.6)) using algorithms based on (24.7). Evaluate these derivatives at the rth estimates $\underline{A} = \underline{A}^{(r)}$, $\underline{B} = \underline{B}^{(r)}$.

(v) From the derivatives in (iv) find the irradiance derivatives following the formulas (24.5), (24.6). Use these to form the $\underline{C}$ matrix in (24.8).

(vi) From the results of (iv), (iii) find the new $\Delta\underline{A}, \Delta\underline{B}$ from (24.9). Label them '$\Delta\underline{A}^{(r+1)}$' and '$\Delta\underline{B}^{(r+1)}$'.

(vii) Define

$$\begin{aligned} \underline{A}^{(r+1)} &\equiv \underline{A}^{(r)} + \Delta\underline{A}^{(r+1)} \\ \underline{B}^{(r+1)} &\equiv \underline{B}^{(r)} + \Delta\underline{B}^{(r+1)} \end{aligned} \tag{24.12}$$

(viii) Make the tests

$$\begin{aligned} |\Delta A_\ell^{(r+1)}/A_\ell^{(r+1)}| &< \varepsilon \\ |\Delta B_\ell^{(r+1)}/B_\ell^{(r+1)}| &< \varepsilon \end{aligned} \tag{24.13}$$

for $\ell = 1,\ldots,m^2$. Here ε is some small number, such as 10^{-3}.

If these $2m^2$ tests are not satisfied, then return to step (i) and repeat the cycle ending with (24.13).

If these $2m^2$ tests are satisfied, then set $\underline{A} \equiv A^{(r+1)}$, $\underline{B} = \underline{B}^{(r+1)}$ and go on to the next subslab beyond X[x,z] to repeat the initialization step D and the preceding seven iteration steps.

Once the A_ℓ and B_ℓ quantities have been found in X[x,z], these are decoded via (24.1) and (24.2) to find a_{jj}, f_{jk}, and b_{jk}. The remaining quantities, ε_f, ε_b, and s_{ij} are found as described in sec. 22E.

The m depth levels $y_1,\ldots,y_m$, at which the m irradiance readings $H(y_u,\pm,j)$, j = 1,...,m are made, are shown in Fig. 8. This set of depths may be shifted downward by some arbitrary amount and the inverse procedure repeated over the new set of depths. For each set of m depths over X[x,z] the $2m^2$ $\underline{\rho}$'s and $\underline{\tau}$'s in (11.17), (11.18) are determined by the above procedure. One visualizes the $\underline{\rho}$'s and $\underline{\tau}$'s, so determined, as average values within the interval X[x,z].

25. FINDING THE INTRINSIC OPTICAL PROPERTIES AND SPECIES CONCENTRATIONS

On the basis of the results in sec. 24, the intrinsic optical properties corresponding to the inherent optical properties $\bar{a}(y,j)$, $\bar{b}(y,j)$, and $\bar{s}(y,i,j)$ (cf. sec. 11) may now be found.

A. *Determining the Specific Absorption Function*

The intrinsic counterpart to the volume absorption function $a(y,\lambda)$, e.g., is defined as follows. Consider an element of volume of the hydrosol at depth y. Suppose a chemical analysis of the solutes and suspensions in the volume yields ℓ uniquely identifiable chemical substances. One may visualize these to be chlorophyll-a, phaeophytins and yellow substances, e.g., so that $\ell = 4$, counting the water molecules as a separate species. The identification of these species is through their absorption spectra over the visible spectrum. Let the concentration (in $\mathrm{mg \cdot m^{-3}}$) of the jth chemical species be $c_j(y)$ at depth y, $x \le y \le z$. Then we postulate the existence of ℓ *specific absorption functions* $a_j(\lambda)$, $\lambda \in \Lambda$, one for each chemical species $j = 1,\ldots,\ell$, with units $\mathrm{m^2 \cdot mg^{-1}}$, such that $a(y,\lambda)$, for all y and λ, is modeled by

$$a(y,\lambda) = \sum_{j=1}^{\ell} c_j(y)\, a_j(\lambda) \tag{25.1}$$

The function $a_j(\lambda)$ of λ is an *intrinsic optical property* of the jth species in the sense that it depends on the molecular structure of the sample and not its mass or volume or location y. The units of the $a_j(\lambda)$ (i.e., $\mathrm{m^2 \cdot mg^{-1}}$) follow from those of $a(y,\lambda)$ (in $\mathrm{m^{-1}}$) and $c_j(y)$ (in $\mathrm{mg \cdot m^{-3}}$). We observe that, while it is correct to write the units of $a_j(\lambda)$ as '$\mathrm{m^{-1}/(mg \cdot m^{-3})}$', the equivalent way '$\mathrm{m^2 \cdot mg^{-1}}$' points up the physical details of the process of absorption, namely that photons are encountering the electrons distributed over the *surfaces* of a unit of mass of the jth molecular species. $a_j(y,\lambda)$ is then a measure of the effective area presented to photons by a unit mass molecular species j about to absorb the photon.

We will now show that the $a_j(\lambda)$ can in principle be determined from concomitant *in situ* measurements of their concentrations $c_j(y)$ and the light field $\{H(y,\pm,i)\colon\ x \le y \le z;\ i = 1,\dots,m\}$. Recall that our procedure in sec. 24 yields $\bar{a}(y_k,j)$, $j = 1,\dots,m$, i.e., the λ-average of $a(y,\lambda)$ over the jth subinterval of Λ (sec. 11) at depth y_k in $X[x,z]$ (Fig. 5). The experimenter is required to supply the estimates of $c_j(y)$, $j = 1,\dots,\ell$ at a sufficient number of depths y. Just how many samples of $c_j(y)$ are needed, can be estimated as follows. Write (25.1), for a fixed wavelength index i, as

$$\bar{a}(y_k,i) = \sum_{j=1}^{\ell} c_j(y_k)\ a_j(i) \tag{25.2}$$

$$i = 1,\dots,m$$

$$x \le y_k \le z \quad , \quad k = 1,\dots,m$$

Here we have replaced $a(y,\lambda)$ and $a_j(\lambda)$ by their averages over the ith spectral subinterval. It is clear now that there are ℓm unknowns in (25.2), namely $a_j(i)$, $i = 1,\dots,m$; $j = 1,\dots,\ell$. We intend to find the $a_j(i)$ by the least squares technique. We must then sample enough depths y_k, say n of them, so that the nm determined $\bar{a}(y_k,i)$ values exceeds ℓm. Thus we require $n > \ell$. Now, Fig. 8 shows that in each subinterval $X[x,z]$ of $X[a,b]$ we can determine m values of $\bar{a}(y_k,i)$. We therefore need to invert the irradiance model in, say, s such subintervals, so that $\ell < n = sm$. For example, suppose we have decided to determine $a(y,\lambda)$ over $m = 5$ wavelength bands, and to identify $\ell = 4$ chemical species (say, water, chlorophyll-a, phaeophytin, and yellow substance). Then the number of depth determinations of $\bar{a}(y_k,i)$ will be $4 = \ell < n$. Hence since $m = 5$, s in this case need only be 1. Therefore, the $\ell m = 4\times 5 = 20$ unknowns $\bar{a}_j(i)$ can be determined by the least squares technique using the $m^2 = 25$ observed values of $\bar{a}(y_k,i)$, $k = 1,\ldots,m$; $i = 1,\ldots,m$, and the $\ell m = 20$ determinations of the concentration $c_j(y_k)$.

The least squares procedure may be formulated for numerical work by placing (25.2) into vector form. Thus, writing n copies of (25.2) for a fixed wavelength interval i, we obtain

$$\begin{bmatrix} \bar{a}(y_1,i) \\ \bar{a}(y_2,i) \\ \vdots \\ \bar{a}(y_n,i) \end{bmatrix} = \sum_{j=1}^{\ell} \begin{bmatrix} c_j(y_1) \\ c_j(y_2) \\ \vdots \\ c_j(y_n) \end{bmatrix} a_j(i) \tag{25.3}$$

write $\quad$ '$\underline{\bar{a}}(\cdot,j)$' for $[\bar{a}(y_1,i),\ldots,\bar{a}(y_n,i)]^T$, $i = 1,\ldots,m$

'$\underline{a}(i)$' for $[a_1(i),\ldots,a_\ell(i)]^T$, $i = 1,\ldots,m$

'$\underline{c}_j$' for $[c_j(y_1),\ldots,c_j(y_n)]^T$, $j = 1,\ldots,\ell$

where 'T' denotes transpose. Then (25.3) becomes

$$\underline{\bar{a}}(\cdot,i) = \underline{C}\ \underline{a}(i) \quad , \quad i = 1,\ldots,m \tag{25.4}$$

where $\quad \underline{C} \equiv [\underline{c}_1 \ \cdots \ \underline{c}_\ell]$

Here $\underline{C}$ is an $n\times\ell$ matrix of measured concentrations, $\underline{\bar{a}}(\cdot,i)$ is the $n\times 1$ vector of average volume absorption coefficient values (of units m^{-1}) for the ith wavelength band found in sec. 24, and $\underline{a}(i)$ is the desired $\ell\times 1$ vector of specific absorption values (of units $m^2\cdot mg^{-1}$). The least squares procedure forms the $\ell\times\ell$ matrix* $(\underline{C}^T\underline{C})^{-1}$ and determines the estimate of $\underline{a}(i)$ by

$$\boxed{\begin{array}{l}\underline{a}(i) = (\underline{C}^T\underline{C})^{-1}\,\underline{C}^T\,\bar{\underline{a}}(\cdot,i)\\ i = 1,\ldots,m\end{array}} \tag{25.5}$$

Thus we determine $\underline{a}(i) = [a_1(i),\ldots,a_\ell(i)]^T$ for $i = 1,\ldots,m$, i.e., all ℓm unknown specific absorption coefficients. Thus we find a set of m spectral absorption values for each of the ℓ molecular species.

B. *Determining Concentrations*

Once the specific absorption functions $a_j(i)$ have been determined we may say that we have *trained* the model (25.2). We may then use the model (25.2) in subsequent experiments to determine the concentrations $c_j(y)$ from the $\bar{a}(y_k,i)$ values yielded by the inverse procedure of sec. 24.

In (25.2) hold depth y_k fixed and write ℓ copies of (25.2), one for each $i = 1,\ldots,\ell$:

$$\begin{bmatrix}\bar{a}(y_k,1)\\ \bar{a}(y_k,2)\\ \vdots\\ \bar{a}(y_k,\ell)\end{bmatrix} = \sum_{j=1}^{\ell} c_j(y_k)\begin{bmatrix}a_j(1)\\ a_j(2)\\ \vdots\\ a_j(\ell)\end{bmatrix} \tag{25.6}$$

$k = 1,\ldots,n$

* One can facilitate the numerical inversion of $\underline{C}^T\underline{C}$ by making a principal component analysis of the set of ℓ column vectors $\{\underline{c}_j:\ j = 1,\ldots,\ell\}$ composing $\underline{C}$. The principal components of $\underline{C}$ form a new set of orthonormal vectors that can replace $\underline{C}$ in (25.3).

This is of course possible only if $\ell \le m$; and we shall adopt this assumption in order to proceed. Writing

'$\bar{a}(y_k,\cdot)$'	for	$[\bar{a}(y_k,1),\ldots,\bar{a}(y_k,\ell)]^T$,	$k = 1,\ldots,n$
'$\underline{a}_j$'	for	$[a_j(i),\ldots,a_j(\ell)]^T$,	$j = 1,\ldots,\ell$
'$\underline{c}(y_k)$'	for	$[c_1(y_k),\ldots,c_\ell(y_k)]^T$	$k = 1,\ldots,n$

we may recast (25.6) into the form

$$\bar{a}(y_k,\cdot) = \sum_{j=1}^{\ell} \underline{a}_j \, c_j(y_k)$$

$$= [\underline{a}_1 \; \underline{a}_2 \; \cdots \; \underline{a}_\ell] \; \underline{c}(y_k)$$

i.e.,

$$\bar{a}(y_k,\cdot) = \underline{A} \; \underline{c}(y_k) \tag{25.7}$$

where $\underline{A} \equiv [\underline{a}_1 \; \cdots \; \underline{a}_\ell]$

Now assuming the $\ell\times\ell$ matrix $\underline{A}$ has rank ℓ, we find

$$\boxed{\begin{array}{l} \underline{c}(y_k) = \underline{A}^{-1} \; \bar{a}(y_k,\cdot) \\ k = 1,\ldots,n \end{array}} \tag{25.8}$$

In this way we obtain the ℓ concentrations $c_1(y_k),\ldots,c_\ell(y_k)$ at depth y_k.

The requirement $\ell \le m$ in (25.6) is necessary in order to obtain a square matrix $\underline{A}$, which is then potentially invertible. The matrix $\underline{A}$ is invertible if the ℓ specific absorption curves, formed by plotting the numbers

$\{a_j(i)\colon\ i = 1,\ldots,\ell\}$ for each $j = 1,\ldots,\ell$, are linearly independent. Graphically, this means that their shapes are to be sufficiently dissimilar. Notice that there is some room for maneuvering here: we need only ℓ of the m possible wavelengths in the discrete spectrum of each molecular species. Suppose, for example, that $(i_1,\ldots,i_\ell)$ is a set of ℓ distinct integers drawn from the integer set $(1,\ldots,m)$. Then form the ℓ wavelength dependent specific absorption curves $\{a_j(i_u)\colon\ u = 1,\ldots,\ell\}$, $j = 1,\ldots,\ell$ and examine them for linear independence--i.e., find the determinant of the matrix $\underline{A}$ formed from these ℓ arrays of ℓ numbers. The larger the determinant's absolute magnitude, the better conditioned will $\underline{A}$ be for inversion in (25.8).

C. *Inversions for* $\bar{b}$ *and* $\bar{s}$

The basic model for $a(y,\lambda)$, namely (25.2) can be written down also for the mean backscatter function $\bar{b}(y,\lambda)$:

$$\bar{b}(y_k,i) = \sum_{j=1}^{\ell} c_j(y_k)\, b_j(i) \tag{25.9}$$

$$i = 1,\ldots,m$$

$$x \le y_k \le z \quad , \quad k = 1,\ldots,m$$

Similarly we can write down

$$\bar{s}(y_k,i',i) = \sum_{j=1}^{\ell} c_j(y_k)\, s_j(i',i) \tag{25.10}$$

$$i',i = 1,\ldots,m$$

$$x \le y_k \le z \quad , \quad k = 1,\ldots,m$$

It follows that the algebraic procedures for inverse procedures to determine the $b_j(i)$ and $s_j(i',i)$ from the light field are precisely those described above for $a_j(i)$. Observe finally that the concentrations $c_j(y_k)$ are common to all three models (25.2), (25.9), (25.10).

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