Chapter 3: THE TWO-PLUS-ONE RUNUP MODEL

This chapter describes the numerical model of the 2+1 propagation and runup of long waves on beaches with arbitrary topography. As earlier, the field equations are the shallow-water wave equations, without friction factors or artificial viscosity. In this chapter, we will describe the numerical algorithm and the validation of the model. Results from the laboratory experiment of the wave runup on a conical island are used as one test for the model verification.

3.1 Introduction

The finite-difference method has been widely used for tsunami propagation modeling and it has become a natural choice for several tsunami inundation models. Shuto (1985, 1991) used the staggered explicit leap-frog finite-difference scheme to simulate the inundation of the 1983 Japan Sea tsunami. This model has been refined by Imamura (1995) and Takahashi (1995) to develop what is now the most widely used model for the tsunami inundation. Several numerical runup models were produced based on the same technique (Liu,1995; Ortiz). Takahashi presented the model on the International Long Wave Runup Workshop to simulate the Hokkaido-Nansei-Oki tsunami, which was one of the four benchmark problems (See description of the workshop on page 8). The inundation calculations produced fairly correct results on the lee of the island, but on the front, they differed by a factor of two from the field data.

The other often used method to solve the shallow-water wave equations the is the finite-element technique. The advantage of the method is the usage of an adaptive non-structured numerical grid for the computations, that makes possible the resolution of the changing wave length of the tsunami with enough grid point throughout the computational domain. The disadvantage is a rather complex and computationally expensive way of inverting the step matrix, as compared with the finite-difference method. Baptista (1995) used the finite-element method to model the benchmark problem of solitary wave runup on a conical island. The results show that the method appears to be very sensitive to changing the model parameters. The model has to be calibrated before the results fit well to the laboratory data. Another finite-element inundation model presented recently is that of Kawahara and Takagi (1995) to solve the same benchmark problem. The model showed a good agreement with the laboratory data. Unfortunately, the authors did not use the method to model the Hokkaido-Nansei-Oki tsunami and one cannot yet comment on the ability to handle simulation over realistic bathymetries.

The goal of this study is to develop a robust numerical method for modeling the interaction of 3-D long waves with nearshore bathymetry and on-shore topography with specific application to tsunami inundation modeling. The proposed model is the shallow-water-wave equations, which is practically the only choice for most large-scale inundation computations at the present state-of-the-science. Despite of its limitations, it will be demonstrated in this study, that it simulates most of the important long wave characteristics during the long-wave runup well enough for most engineering applications.

As a part of the effort to reduce parameters of the model, no bottom friction terms are included in this model. Although the bottom friction does affect the dynamic of the run-up process in the surf zone, several reasons were considered for not using the friction terms in the model. The commonly used bottom friction model for shallow-water-wave approximation is the Chésy formula with different types of roughness coefficient (Packwood and Peregrine, 1981; Mader,1984; Kowalik,1987; Liu,1995; Kobayashi et al 1987, etc.). This formula is an empirical relationship developed from steady channel flows, so it might not reflect the dynamic of the rapid runup process adequately. Also, there is no consensus on a proper form of the roughness coefficient in the formula. A number of studies are devoted to the designing of a proper roughness coefficient instead of the commonly used Manning's coefficient (Fujima and Shuto, 1989). On the other hand, several studies show that an unsteady flow during runup is not very sensitive to changes in the roughness coefficient value (Pakwood and Peregrine 1981, Kobayashi et al 1987). Any numerical algorithm of moving boundary for the wave runup induces a numerical friction near the tip of the climbing wave (except perhaps, a Lagrangian formulation). This complicates the proper choice of the friction coefficient for any numerical model. The roughness coefficient in the numerical model at present appears a quiet arbitrary parameter that is adjusted to fit a given experimental data, but is very difficult to be determined a priori. Given that extrapolation is always visky, this reduces dramatically the prediction ability of a numerical model with ad-hoc parameters. Since the goal is the evaluation of the maximum runup level and the maximum inundation velocities which the friction can only reduce, it was decided not to complicate the model with additional adjustment parameter.

The solution introduced here uses the finite-difference method for solving the non-linear shallow-water-wave equations, based on the method of fractional steps (Yanenko, 1971). This method hereafter referred to as VTCS-3 is the three-dimensional extension of the 1+1 solution of Titov and Synolakis (1995) that has been extensively tested for the two-dimensional case. The ability of the model to reproduce laboratory data for 3-D waves, and to simulate various features of the real tsunamis will be demonstrated in the following sections.

3.2 Numerical model

3.2.1 Mathematical formulation

We use the two-dimensional shallow-water-wave equations (SW) to model the long wave generation, propagation, evolution over physical bathymetry and runup phenomenon. The 2+1 shallow-water-wave equations are

$$\begin{aligned} h_t + (uh)_x + (vh)_y &= 0 \\ u_t + uu_x + vu_y + gh_x &= gd_x \\ v_t + uv_x + vv_y + gh_y &= gd_y \end{aligned} \tag{3.23}$$

where $h = \eta\,(x,y,t) + d(x,y,t)$, $\eta\,(x,y,t)$ is the amplitude, $d(x,y,t)$ is the undisturbed water depth, $u(x,y,t)$, $v(x,y,t)$ are the depth-averaged velocities in the x and y directions respectively, g is the acceleration of gravity.

A variety of boundary and initial conditions can be specified for these equations. To solve the problem of tsunami generation due to bottom displacement, the following initial conditions are specified

$$
\begin{aligned}
d(x, y, t) &= d_0(x, y, t),\ t \le t_0 \\
d(x, y, t) &= d_0(x, y, t_0),\ t \le t_0
\end{aligned}
\tag{3.24}
$$

Usually, t_0 is assumed to be small, so that the bottom movement is an almost instantaneous vertical displacement, i.e. the initial condition is not time-dependent.

To produce waves entering into the computational area through the boundary $y = y_b$ the following boundary conditions are specified

$$
\begin{aligned}
v(x, y, t) &= v_0(x, y_b, t) \\
h(x, y, t) &= h_0(x, y_b, t)
\end{aligned}
\tag{3.25}
$$

The proper boundary conditions should be specified for open-sea boundary and for land boundary. These conditions will be discuss later in this chapter.

3.2.2 Splitting technique

For arbitrary topography and bottom displacement the system of equations (3.23) has to be solved numerically. Consider the finite-differences algorithm based on the splitting method, i. e. the method of fractional steps of Yanenko (1971). This method reduces the numerical solution of the two-dimensional problem into consecutive solution of two instantaneous one-dimensional problems. This is achieved by splitting the governing system of equations (3.23) (page 51) into a pair of systems, each containing only one space variable, as follows

$$
\left\{
\begin{aligned}
& h_t + (uh)_x = 0 \\
& u_t + uu_x + gh_x = gd_x \\
& v_t + uv_x = 0
\end{aligned}
\right\}
\quad \text{and} \quad
\left\{
\begin{aligned}
& h_t + vh_y = 0 \\
& v_t + vv_y + gh_y = gd_y \\
& u_t + vu_y = 0
\end{aligned}
\right\}.
\tag{3.26}
$$

The two systems of equations (3.26) can be solved sequentially at each time step using appropriate numerical methods. Here, an explicit finite-difference scheme is used to solve (3.26), although, the conventional procedure for the splitting method is to use implicit numerical scheme for solving 1+1 equations. The implicit method works effectively for elliptic and parabolic equations, where splitting provides a substantial reduction of the number of operations compared with application of the implicit scheme directly to 2+1 elliptic or parabolic equations (Fletcher, 1991, Ch. 8.5). In this case system (3.23) is a hyperbolic quasi-linear system, where explicit methods have proven to be very efficient. Here, it was found advantageous to use the splitting method in combination with an explicit finite-difference technique. The main advantage of this approach is the use a characteristic form of the 1+1 equations. The characteristic analysis helps establish a well-posed boundary-value problem. The characteristic form of equations also allows for an efficient finite-difference realization (Titov and Synolakis, 1995).

Each of the systems (3.26) is a hyperbolic quasi-linear system with all three real and distinct eigenvalues. It can be written in characteristic form as follows,

$$\begin{aligned} p_1 + \lambda_1 p_x &= g d_x \\ q_1 + \lambda_1 q_x &= g d_x \\ v'_1 + \lambda_1 v'_x &= 0 \end{aligned} \tag{3.27}$$

where

$$\begin{aligned} p &= u + 2\sqrt{gh} \\ q &= u - 2\sqrt{gh} \\ v' &= v \end{aligned} \tag{3.28}$$

are the Riemann invariants of this system and

$$
\begin{aligned}
\lambda_1 &= u + \sqrt{gd} \\
\lambda_2 &= u - \sqrt{gd} \\
\lambda_3 &= u
\end{aligned}
\tag{3.29}
$$

are the eigenvalues.

An explicit finite-difference method is used to solve equations (3.27) along x and y coordinate sequentially at every time step. The first two equations in (3.27) constitute a one-dimensional shallow-water wave evolution problem. It means that every time step a one-dimensional long wave propagation problem is solved along each coordinate plus one more equation describing a nonlinear momentum flux in the direction normal to the coordinate. The method developed for the 1+1 long wave propagation and runup (Titov and Synolakis, 1995) is used to solve each of the systems (3.27).

The overall procedure utilizing the splitting technique for the system (3.23) can be summarized as follows. Given u^n, v'^n, h^n at time t, the algorithm of computing values u^{n+1}, v'^{n+1}, h^{n+1} for time instant $t+\Delta t$ involves the following steps.

1. Convert the primitive variables u^n, v^n, h^n into the Riemann invariants p^n, q^n, v'^n using the transformation (3.28).
2. Compute values $p^{n+1/2}$, $q^{n+1/2}$, $v'^{n+1/2}$ by solving numerically system (3.27) along the x-coordinate.
3. Convert $p^{n+1/2}$, $q^{n+1/2}$, $v'^{n+1/2}$ to the primitive variables $u^{n+1/2}$, $v^{n+1/2}$, $h^{n+1/2}$ using the transformation inverse to (3.28).
4. Repeat the steps one through three above for $u^{n+1/2}$, $v'^{n+1/2}$, $h^{n+1/2}$ along the y-coordinate to compute the values u^{n+1}, v'^{n+1}, h^{n+1}. Note, that the Riemann invariants are different during that step, because u and v are interchanged in (3.28) and (3.29).

3.2.3 Boundary conditions for fixed boundaries

The splitting method requires the solution of one-dimensional systems of equations (3.26) every time step. Therefore, boundary conditions should be established for the two 1+1 problems. As in chapter 2, characteristic analysis will be used to establish a well-posed boundary values for a one-dimensional hyperbolic system. The following discussion uses the method of section 2.3.2 which is applied here for the system (3.27).

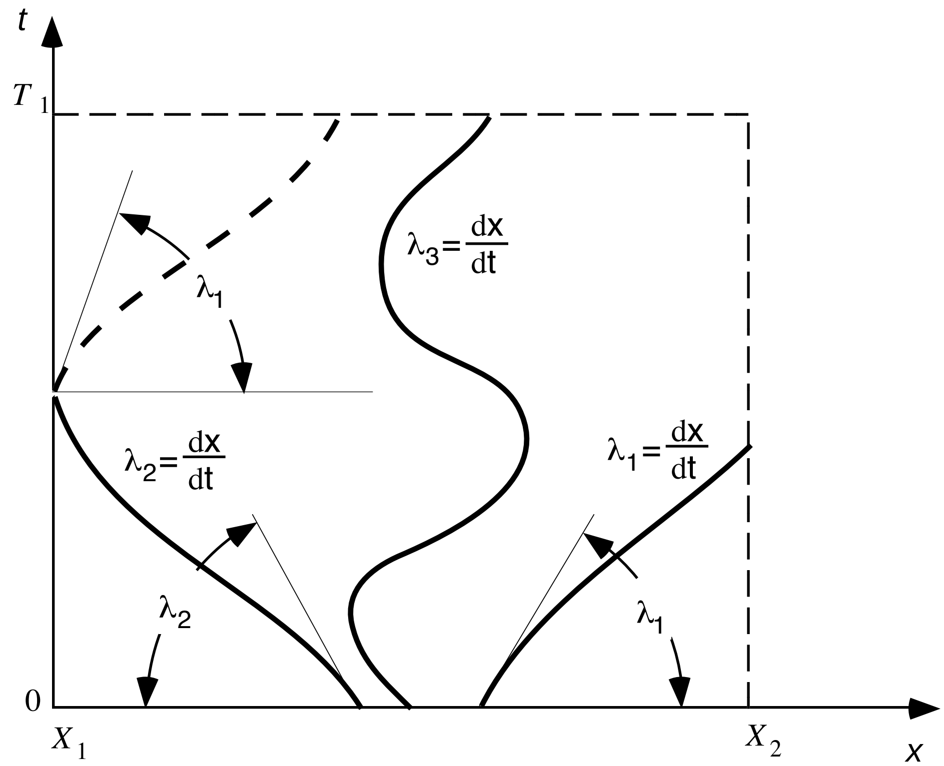

Figure 3.16 Sketch of the characteristic lines for the system (3.27).

Consider solution of the system of equations (3.27) in the area shown in Figure 3.16. It is a hyperbolic quasi-linear system with all real and different eigenvalues (3.29). It has three families of characteristic lines with slopes λ_1, λ_2 and λ_3; The eigenvalue λ_1 is positive while λ_2 is negative everywhere in the region, where the Froude number is less then

1; λ_3 can be either positive or negative. A well-posed boundary-value problem requires the number of boundary conditions for the Riemann invariants to be equal to the number of outgoing characteristic lines for this boundary. Therefore, one or two boundary conditions are necessary on each boundary depending on the sign of λ_3 on the boundary.

The boundary conditions for a totally reflective boundary $x = X_1$ are

$$\begin{aligned} p &= -q \\ v' &= 0 \end{aligned} \tag{3.30}$$

while the reflective conditions for the boundary $x = X_2$ are

$$\begin{aligned} q &= -p \\ v' &= 0 \end{aligned} \tag{3.31}$$

As in chapter 2, the method of Gustafsson and Kreiss (1979) is used to develop an absorbing boundary conditions for time dependent problems. A totally absorbing boundary allows waves to go through (absorb) but it does not allow any waves to reflect back into the computation region. In characteristic terms, the invariants on outgoing characteristics do not carry any disturbances back into the computational area. For the boundary $x = X_2$, the requirement of no wave motion on these characteristics implies that $u = 0$, $v = 0$, $\eta = 0$, then $q = -2\sqrt{gd(X_2)}$, and $v' = 0$. In addition, it is assumed that the water depth is constant outside the area of computation and equal to the depth at the right boundary $d(X_2)$; then equation (3.27) implies that q is constant on that boundary. Therefore, the appropriate conditions are

$$\begin{aligned} q &= -2\sqrt{gd(X_2)} \\ v' &= 0 \end{aligned} \tag{3.32}$$

For the left boundary the absorbing conditions are

$$\begin{aligned} p &= 2\sqrt{gd(X_1)} \\ v' &= 0 \end{aligned} \tag{3.33}$$

The runup computations require a moving boundary conditions to be used for the tip of the shoaling wave. Titov and Synolakis (1995) developed the moving boundary condition for one-dimensional shallow-water-wave equations, which is described in chapter 2. The same basic approach can be used for the runup boundary condition for the system (3.27).

3.2.4 Finite-difference scheme

To solve each equation in the system (3.27) the same explicit finite-difference scheme (2.7) developed for the 1+1 problem is used. The scheme allows for the spatial grids with a variable space steps Δx_i. The condition of the stability for this scheme is the Courant-Friedrichs-Lewy criterion

$$\Delta t \le min\frac{\Delta x_i}{|u_i| + \sqrt{gh_i}} \tag{3.34}$$

The finite-difference scheme (2.7) is used for the computation of the unknown variables p, q and v' in the interior grid points of the computational area. However, these equations can't be used to compute boundary values. At those points, the boundary conditions (3.30) – (3.33) determine only two among the three invariants. The other value on the boundary (the value of the Riemann invariant on the incoming characteristic) is computed using one of the governing equations (3.27) by the upwind finite-difference scheme

$$p_b^{n+1} = p_b^n - \frac{\Delta t}{\Delta x}[\lambda_1^n(\Delta_{-x} p_b^n) - g(\Delta_{-x} d_b^n)] \tag{3.35}$$

where p_b, d_b are the values of the variables on the boundary.

Inundation computations use moving boundary conditions to calculate the evolution of the wave on the dry bed. Since the 2+1 splitting technique uses the same finite-difference scheme as the 1+1 solution, the moving boundary algorithm is exactly the same as described in Chapter 2 on page 27. The only addition to the 1+1 algorithm is that the velocity component normal to the direction of computation is kept zero on the moving shoreline point.

During shoaling the wavelength of tsunamis becomes shorter. Therefore calculations using a uniform grid throughout the computational domain suffer either loss of accuracy in the nearshore field or loss of efficiency if a very fine grid is used. Either approach does not produce consistent resolution. Here a variable grid in each direction is used. The variable grid can create a consistent resolution for a one-dimensional domain (Titov and Synolakis, 1995), or for a cylindrical two-dimensional domain, when the depth is changing predominantly along one direction (Titov, 1989). To model tsunami wave propagation in areas with complex bottom profiles containing complicated shoreline patterns and islands, an additional nested grid is used for the near-shore computations. The nested grid has finer grid–spacing for an efficient computation of the shorter waves in the nearshore area. A special algorithm has been developed to automatically create a nested grid for the areas where depth is less than a certain value. The same numerical scheme (2.7) (page 23) computes the wave field on both grids at the same time. The computed values are interpolated on the boundaries between the grids every time step to provide a continuous flow of information between the areas. This approach allows for a computation in a large complicated areas with a minimum loss of accuracy due to inconsistent resolution of the finite-difference grid.