A Path-Conservative Finite Volume Method for Numerical Simulation of Tsunami Waves with Evolving Bathymetry

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Introduction and model

Goal

This work presents a morpho-hydrodynamic model and a numerical approximation designed for the fast and accurate simulation of sediment movement associated with extreme events, such as tsunamis. The model integrates the well-established hydrostatic shallow-water equations with a transport equation for the moving bathymetry, relying on a bedload transport function. Subsequently, this model is discretized using the path-conservative finite volume framework to yield a numerical scheme that is not only fast but also second-order accurate and well-balanced for the lake-at-rest solution. The numerical discretization separates the hydrodynamic and morphodynamic components of the model but leverages the eigenstructure information to evolve the morphologic part in an upwind fashion, preventing spurious oscillations. The study includes various numerical experiments, incorporating comparisons with laboratory experimental data and field surveys. The final code is accurate and very fast, thanks to the GPU parallelization implementation.

Model Equations

The two dimensional, $\boldsymbol{x} = (x, y)$, system of partial differential equations evolving in time t reads,

$$\begin{cases} \partial_t h + \nabla_x (hu) = 0, \\ \partial_t (hu) + \nabla_x \left(hu \otimes u + g\dfrac{h^2}{2} \right) = gh\nabla_x H - ghS_f, \\ \partial_t H - \nabla_x q_b = 0. \end{cases} \tag{1}$$

Here, h is the total water column, $u = (u_x, u_y)$ is the horizontal velocity vector, g stands for the gravity, H is the bathymetry function and S_f represents the friction terms. Of particular interest is q_b, that controls bedload sediment transport rate per unit time. Finally, $\nabla_x = (\partial_x, \partial_y)$ represent the two dimensional spatial derivative operator.

Note that system (1) is strictly hyperbolic with Darcy-Weisbach's friction formula.

Hyperbolicity

The eigenvalues of the one dimension system are:

$$\begin{cases} \lambda_1 = 2\sqrt{-S}\, cos(\theta/3) - a_1/3 \\ \lambda_2 = 2\sqrt{-S}\, cos((\theta + 2\pi)/3) - a_1/3 \\ \lambda_2 = 2\sqrt{-S}\, cos((\theta + 4\pi)/3) - a_1/3 \end{cases} \tag{2}$$

with $S = (3a_2 - a_1^2)/9$, $\theta = arcos(R/\sqrt{-S^3})$ and $R = (9a_1a_2 - 27a_3 + 2a_1^3)/54$. Also,

$$a_1 = -2\frac{q}{h}, \qquad a_2 = \frac{q^2}{h^2} - gh\left(1 + \frac{\partial q_b}{\partial q}\right), \qquad a_3 = -gh\frac{\partial q_b}{\partial h}.$$

Conversely, recall the simple eigenstructure of the standard shallow water equations:

$$\lambda_1 = u + \sqrt{gh}, \qquad \lambda_2 = u - \sqrt{gh}. \tag{3}$$

Discharge formula

The bedload transport rate is given by,

$$q_b = \frac{8Q}{1-\phi} sgn(S_f)(\theta - \theta_{crit})_+^{3/2} \tag{4}$$

with Q the characteristic discharge and S_f the Darcy-Weisbach's friction formula:

$$Q = \sqrt{\left(\frac{\rho_s}{\rho} - 1\right) g d_s^3}, \qquad S_f = \frac{fu|u|}{8gh}.$$

Note that the q_b depends on the friction coefficient, but also of Q, ϕ, θ or θ_{crit}. These parameters are given by the type of sediment and depend on the ratio of the drag forces and submerged weight (θ), the porosity of the sediment layer (ϕ), the sediment density and mean diameter (Q) or a friction coefficient (S_f).

Numerical implementation

Numerical scheme

System (1) can be written in the form of a general hyperbolic system of conservation laws, with conservative fluxes and non-conservative products as follows:

$$\partial w + \partial_x F_C(w) + P(w, \partial_x \eta) = S(w),$$

where w is the state vector of conserved variables defined by:

$$w = (h, hu, H)^T.$$

and $F(w)$, $P(w, \partial_x\eta)$ and $S(w)$ the conservative flux, the pressure term (non-conservative product) and the source terms respectively.

The second order accurate finite volume numerical scheme reads,

$$w_i'(t) = -\frac{1}{\Delta x}\left(D^+_{i-\frac{1}{2}}(t) + D^-_{i+\frac{1}{2}}(t)\right) - \frac{1}{\Delta x}\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} P(R_i^t, \partial_x R_i^{\eta,t})\, dx. \tag{5}$$

Here, $R_i^{\eta,t}$ denotes the reconstruction operator. Note that the numerical fluxes $D^{\mp}_{i\pm\frac{1}{2}}$ are computed using the reconstructed states:

$$D^-_{i+\frac{1}{2}} = \frac{1}{2}\left((1 - \alpha_{1,i+\frac{1}{2}})E_{i+\frac{1}{2}} - \alpha_{0,i+\frac{1}{2}}(w_{i+1} - w_i)\right) + F_C(w_i).$$

Here $E_{i+\frac{1}{2}} = F_C(w_{i+1}) - F_C(w_i) + P_{i+\frac{1}{2}}$ while $\alpha_{\{0,1\},i+\frac{1}{2}}$ is an approximation of the maximum and minimum wave speed of the system. We use the shallow water eigenvalues (3) to compute them. For evolving the bathymetry variable, the following upwind method is considered, with the following approximation for the wave speed, $\beta_{0,i+\frac{1}{2}}$, $\beta_{1,i+\frac{1}{2}}$:

$$\beta_{0,i+\frac{1}{2}} = 0$$

$$\beta_{1,i+\frac{1}{2}} = \begin{cases} \alpha_{1,i+\frac{1}{2}} & \text{if } sgn(\alpha_{1,i+\frac{1}{2}}) = \frac{sgn(F_C(H_{i+1}) - F_C(H_i))}{sgn(H_{i+1} - H_i)} \\ -\alpha_{1,i+\frac{1}{2}} & \text{otherwise.} \end{cases}$$

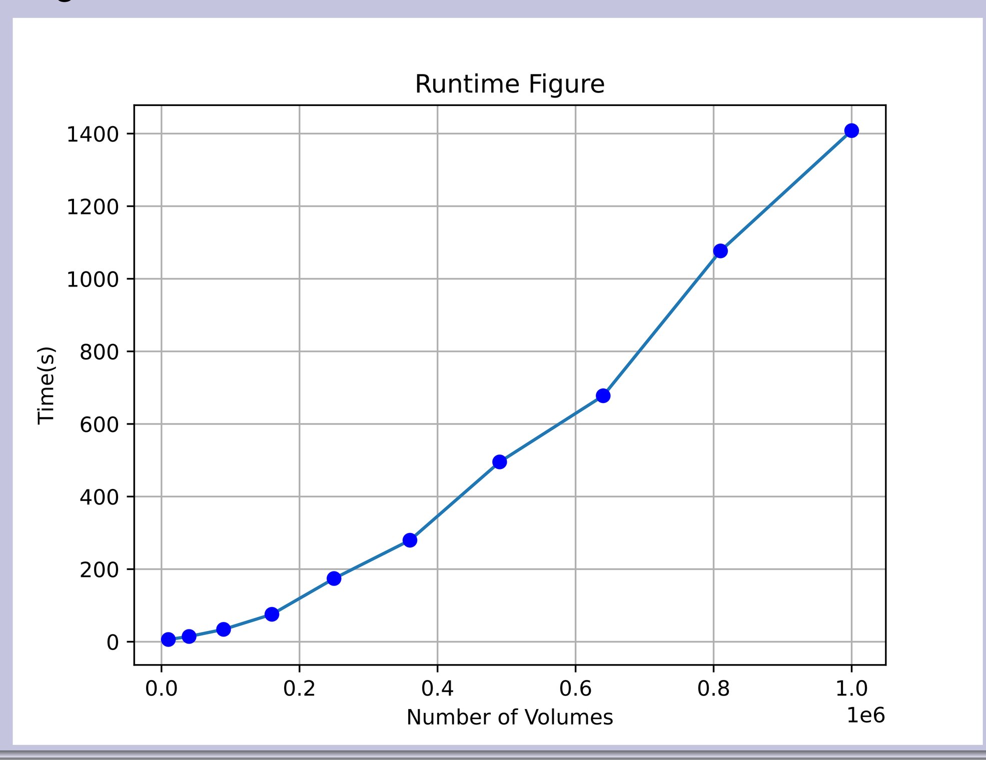
Numerical scheme characteristics

Some properties of the model:

- Second order in space is achieved by a MUSCL reconstruction.
- A second order TVD Runge-Kutta is used to accomplish second order in time.
- A hydrostatic reconstruction is also performed, guaranteeing the well-balanced property for the water at rest solution.
- Positive preserving with suitable CFL condition.
- No spurious oscillation related with the splitting of the hydrodynamic and morphodynamic part.

GPU parallelization

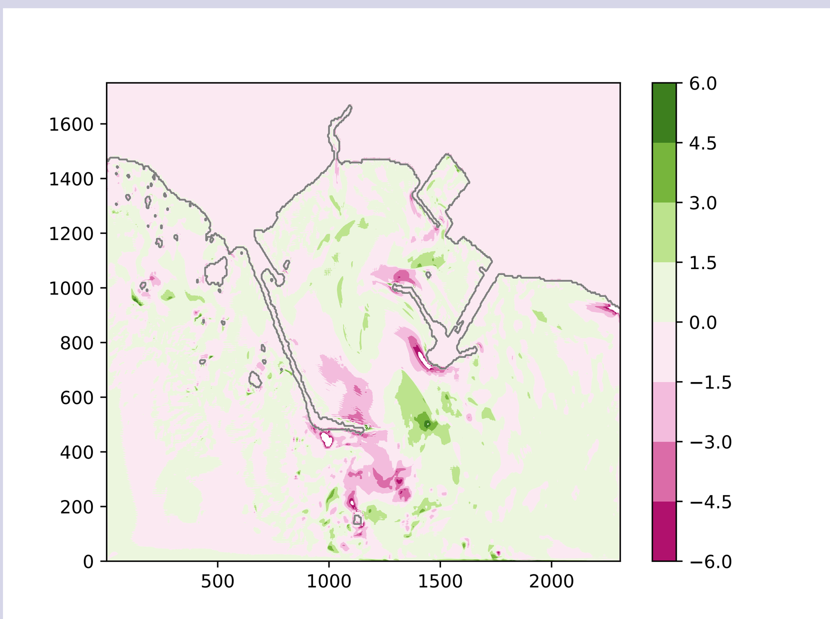
The model has been parallelized in GPU using CUDA for speed. The following figure depicts some computation time in a single Nvidia Tesla V100 GPU and Intel Xeon Silver 4214 CPU @ 2.20GHz,



Numerical results

Impact of Tōhoku 2011 tsunami event in Crescent City harbor

This problem address morphologycal changes in Crescent City (California, U.S.) harbor after the 2011 Tōhoku tsunami event. We compare our numerical results with field survey data pre and post event.



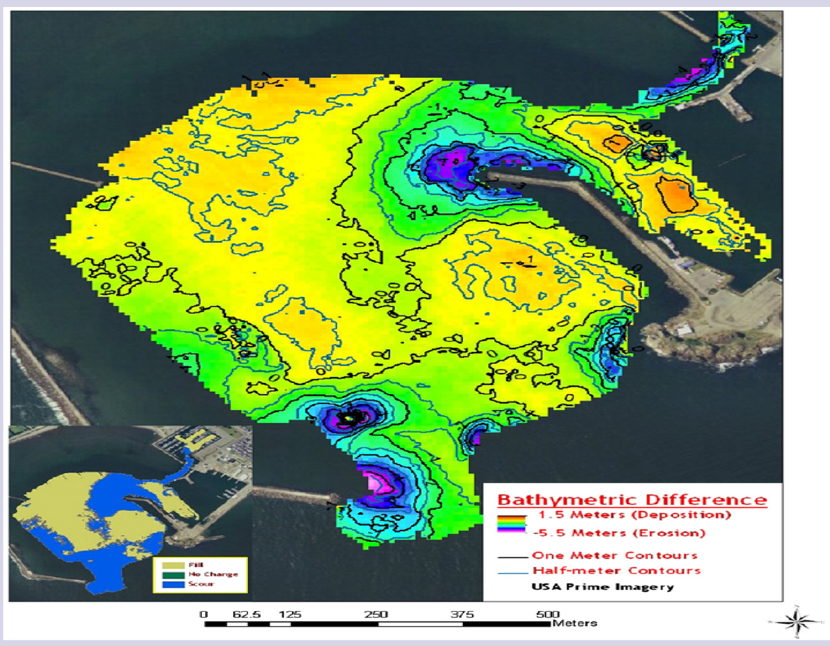
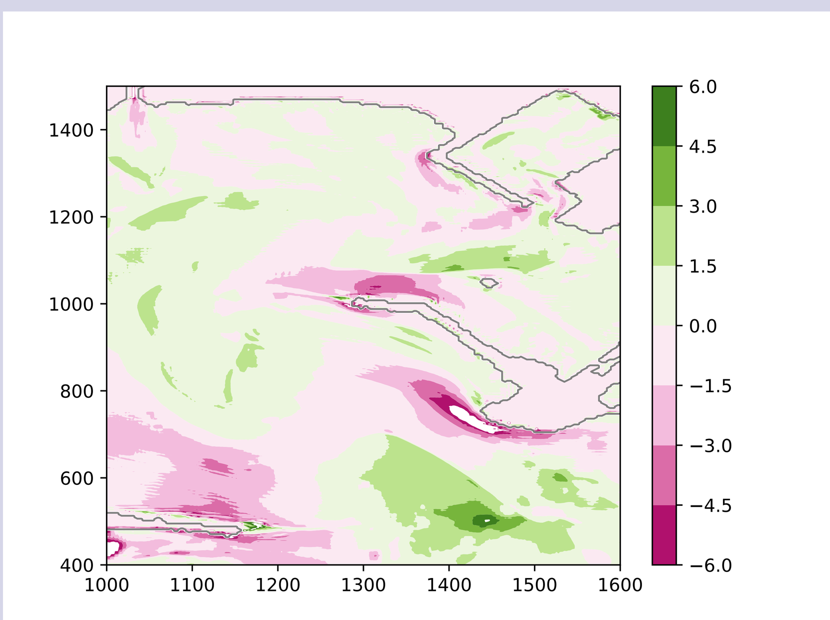


Figure: Difference in bathymetry after the tsunami event. Numerical results (**left**) and Wilson *et. al.* survey results (**right**).



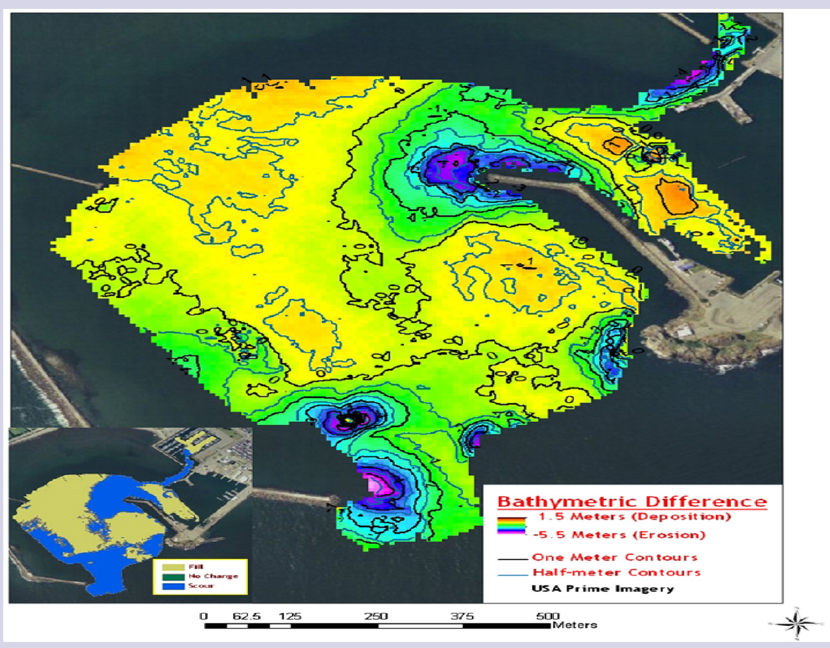
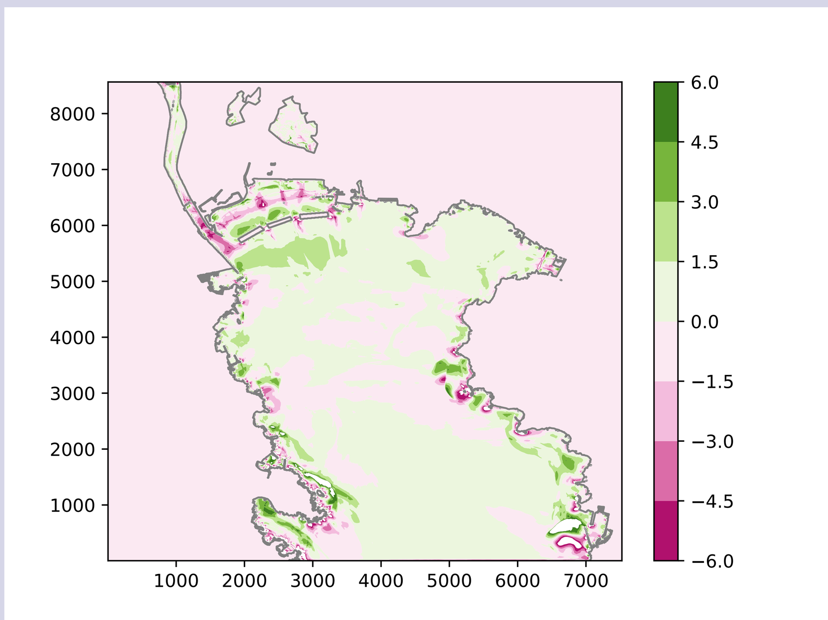


Figure: Difference in bathymetry after the tsunami event. Numerical results zoom (**left**) and Wilson *et. al.* survey results (**right**).

Impact of Tōhoku 2011 tsunami event in Hirota Bay

The final experiment concerns the impact of the 2011 Tōhoku tsunami event in Hirota Bay, in Rikuzentakata City, Japan. Again, numerical results are compared against field data survey.



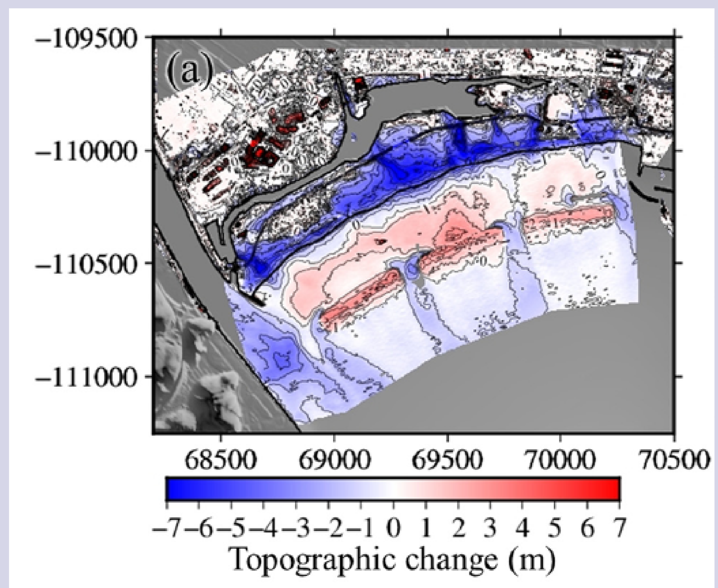
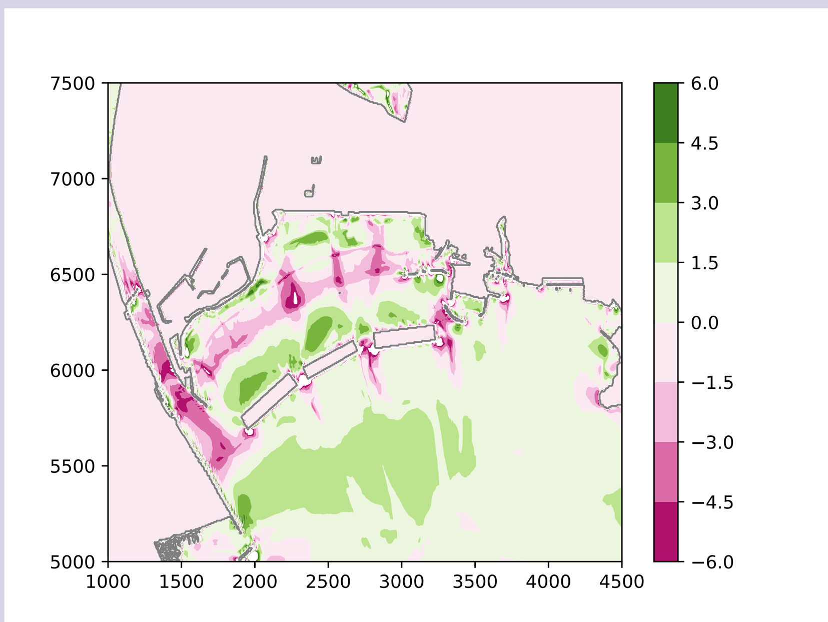


Figure: Difference in bathymetry after the tsunami event. Numerical results zoom (**left**) and field survey results (**right**).



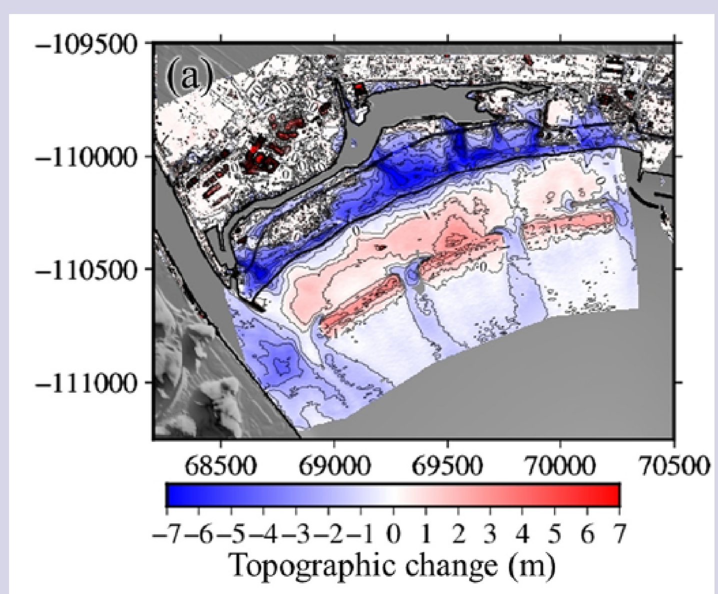


Figure: Difference in bathymetry after the tsunami event. Numerical results zoom (**left**) and field survey results (**right**).

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Bibliography

[1] M. Castro Díaz, E. Fernández–Nieto, and A. Ferreiro. "Sediment transport models in Shallow Water equations and numerical approach by high order finite volume methods". In: *Computers & Fluids* 37.3 (Mar. 2008), pp. 299–316.